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Using congruence relations to extract knowledge from concept lattices

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ABSTRACT

It is well-known inside the Formal Concept Analysis (FCA) community that a concept lattice could have an exponential size with respect to the input data. Hence, the size of concept lattices is a critical issue in large real-life data sets. In this paper, we propose to investigate congruence relations as a tool to get meaningful parts of the whole lattice or its implication basis. This paper presents two main theoretical contributions, namely two context (or lattice) decompositions based on congruence relations and new results about implication computation after decomposition.

1. Introduction

1.1. General overview

During the last decade, the computation capabilities have promoted Formal Concept Analysis (FCA) with new methods based on concept lattices. Though they are exponential in space/time in worst case, concept lattices of a reasonable size enable an intuitive representation of data expressed by a formal context that links objects to attributes through a binary relation. Methods based on concept lattices have been developed in various domains such as knowledge discovery and management, databases or information retrieval where some relevant concepts, *i.e.* possible correspondences between objects and attributes are considered either as classifiers, clusters or representative object/attribute subsets.

With the increasing size of data, a set of methods have been proposed in order to either generate a subset (rather than the whole set) of concepts and their neighborhood in an on-line and interactive way [14,37] or better display lattices using nested line diagrams [21]. Such approaches become inefficient when contexts are huge. However, the main idea of lattice/context decomposition into smaller ones is still relevant when the classification property of the initial lattice is maintained. Many lattice decompositions have been defined and studied, either from an algebraic point of view [12,28] or from an FCA point of view [21,19]. We can cite the Unique Factorization Theorem [28], the matrix decomposition [4], the Atlas decomposition [21], the subtensorial decomposition [21], the subdirect decomposition [12,17,15,16,38–41,19], or the

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doubling convex construction. The doubling convex construction has also been widely studied [7,10,29,22,5], mainly from a theoretical point of view in order to characterize lattices that can be obtained by such decomposition.

FCA is also widely used as a framework for association rule mining which is an important topic in the data mining research area. As far as we know, no links between decompositions cited above and implication computation have been made. However, we note that [33] tackles a similar issue in the case of a vertical decomposition of contexts into subcontexts.

In this paper, we present a synthesis about previous studies on the subdirect decomposition [35] and the reverse doubling construction [36], both based on congruence relations. To go further, we investigate links between implications and congruence relations. New results of this paper are given in Section 4.

Firstly, we investigate the subdirect decomposition of a concept lattice as a step towards an interactive exploration and mining of large contexts. The subdirect decomposition of a lattice L into factor lattices $(L_i)_{i\in\{1,\dots,n\}}$, denoted by $L \hookrightarrow L_1 \times \cdots \times L_n$, is defined by two properties (see important results in [21]): (i) L is a sublattice of the direct product $L_1 \times \cdots \times L_n$, and (ii) each projection of L onto a factor is surjective.

Each factor lattice generated from the decomposition is actually the concept lattice of an arrow-closed subcontext, *i.e.* closed according to the arrow relation between objects and attributes. This means that the decomposition can be obtained by computing specific subcontexts. Moreover, there is an equivalence between arrow-closed subcontexts and congruence relations of L, *i.e.*, an equivalence relation whose equivalence classes form a lattice with elements closed by the meet and join operations. This means that the concepts of L can be retrieved from the factor lattices, and the classification property of L is maintained since each equivalence relation forms a partition of the elements. Another result establishes an equivalence between arrow-closed subcontexts and compatible subcontexts, *i.e.* subcontexts in which each concept corresponds to a concept of the initial lattice. This result gives a way to compute the morphism from L into the direct product, and thus to retrieve the concepts of L from the factor lattices.

It is important to note that the four following notions are equivalent:

- Factors of a subdirect decomposition,
- Congruence relations,
- Arrow-closed subcontexts and
- Compatible subcontexts.

As suggested in [21], the contexts of the factors of a particular subdirect decomposition, namely the subdirectly irreducible subcontexts, can be obtained by a polynomial processing of each row/object of the initial context. Therefore, the subdirect decomposition of a lattice can be extended to a subdirect decomposition of its reduced context into subdirect and irreducible subcontexts.

This decomposition leads to data storage saving of large contexts. Indeed, the generation of the whole set of factor lattices can be avoided by providing an interactive generation of a few (but not all) concepts and their neighborhood from large contexts. Moreover, a focus on a specific factor lattice can be proposed to the user by generating, partially or entirely, the concept lattice and/or a basis of implications.

Secondly, we investigate a method named reverse doubling construction to reduce the size of data. In other words, we propose a method to construct a smaller lattice from a given one. It is based on the previous work of Day about the doubling convex construction [8,7,10]. Such a method has then been generalized [22,31] and further widely studied [11,29,5]. Intuitively, this construction consists in doubling a convex subset C of nodes into a lattice L. In this paper, we propose a "reverse doubling construction" which aims at removing from a lattice L a convex set. However this removal construction has hypothesis, so there may be no convex set to remove and hence no changes to perform on the initial lattice structure. When the reverse doubling construction is successfully applied to an assumed *finite lattice* L and a convex set C, then with Day's doubling construction, the initial lattice L can be recovered without any loss of information.

Studies about the doubling convex construction can be organized according to the following chronological sequence of events:

- The first one corresponds to the original work of Day [7–11,29], who introduced the construction. At the very beginning, only intervals were doubled.
- Further generalizations were developed that lead to the general doubling convex construction [22,31].
- In parallel, characterizations of lattices obtained by iterating the doubling convex construction were investigated [8,10, 11,29,5].
- In this paper, we define a reverse construction which removes a convex set from a lattice whenever it is possible [36].

Being able to recover the full lattice from the smaller one and a convex set means that all the information is contained in the small lattice. Thus we only need to consider the subcontext that defines the small lattice. In other words, only a part of the data is relevant. Consequently, one needs only to access or even keep a smaller part of the data. This is obviously one interesting way, among others, to manage big data that are nowadays ubiquitous in many real-life applications.

This paper is organized as follows. The next subsections give the background needed to understand the rest of the article. The next sections present the subdirect decomposition (Section 2), the reverse doubling construction (Section 3) and some links between implications and congruence relations (Section 4). Conclusion and future work are given in Section 5.

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