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ORIGINAL ARTICLE

The fractional complex transformation for nonlinear fractional partial differential equations in the mathematical physics

Elsayed M.E. Zayed *, Yasser A. Amer, Reham M.A. Shohib

Mathematics Department, Faculty of Sciences, Zagazig University, Zagazig, Egypt

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Abstract In this article, the modified extended tanh-function method is employed to solve fractional partial differential equations in the sense of the modified Riemann–Liouville derivative. Based on a nonlinear fractional complex transformation, certain fractional partial differential equations can be turned into nonlinear ordinary differential equations of integer orders. For illustrating the validity of this method, we apply it to four nonlinear equations namely, the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations, the space–time fractional nonlinear Whitham–Broer–Kaup equations, the space–time fractional nonlinear coupled Burgers equations and the space–time fractional nonlinear coupled mKdV equations.

1. Introduction

Fractional differential equations are the generalizations of classical differential equations with integer orders. In recent years, nonlinear fractional differential equations in mathematical physics are playing a major role in various fields, such as physics, biology, engineering, signal processing, and control theory, finance and fractal dynamics (Miller and Ross, 1993; Kilbas et al., 2006; Podlubny, 1999). Finding approximate and exact solutions to the fractional differential equations is an important task. A large amount of literatures were developed concerning the solutions of the fractional differential equations in nonlinear dynamics (El-sayed et al., 2009). Many powerful and efficient methods have been proposed to obtain the numerical and exact solutions of fractional differential equations. For example, these methods include the variational iteration method (Safari et al., 2009); Wu and Lee, 2010; Yang and Baleanu, 2012; Guo and Mei, 2011), the Lagrange characteristic method (Jumarie, 2006a), the homotopy analysis method (Song and Zhang, 2009), the Adomian decomposition method (El-Sayed and Gaber, 2006; El-sayed et al., 2010), the homotopy perturbation method (He, 1999; He, 2000; Yildirim and Gulkanat, 2010), the differential transformation method (Odibat and Momani, 2008), the finite difference method (Cui, 2009), the finite element method (Huang et al., 2009), the fractional sub-equation method (Zhang and Zhang, 2011; Guo et al., 2012; Lu, 2012), the (G'/G)-expansion method (Zheng, 2012; Gepreel and Omran, 2011; Younis and Zafar, 2013), the modified extended tanh-function method (El-Wakil et al., 2005; El-Wakil et al., 2002; Soliman, 2006; Dai and Wang, 2014), the fractional complex transformation

* Corresponding author.
E-mail addresses: e.m.e.zayed@hotmail.com (E.M.E. Zayed), yaser31270@yahoo.com (Y.A. Amer).
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method (Li and He, 2010; Li, 2010; He and Li, Li et al., 2012; Hristov, 2010), the exp-function method (He, 2013), the similarity transformation method (Dai et al., 2013; Zhu, 2013), the Hirota method (Liu et al., 2013) and so on.

The objective of this paper is to apply the modified extended tanh-function method for solving fractional partial differential equations in the sense of the modified Riemann–Liouville derivative which has been derived by (Jumarie, 2006b). These equations can be reduced into nonlinear ordinary differential equations (ODE) with integer orders using some fractional complex transformations. Jumarie's modified Riemann–Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\eta)^{-\alpha}[f(\eta)-f(0)]d\eta, 0<\alpha\leqslant 1, \\ \left[f^{(n)}(t)\right]^{(\alpha-n)}, n\leqslant\alpha<n+1, n\geqslant 1 \end{cases}$$

We list some important properties for the modified Riemann–Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, r>0 \tag{1}$$

$$D_t^\alpha[f(t)g(t)] = f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t) \tag{2}$$

$$D_t^\alpha[f(g(t))] = f\prime_g(g(t))D_t^\alpha g(t) \tag{3}$$

$$D_t^\alpha[f(g(t))] = D_g^\alpha f(g(t))[g\prime(t)]^\alpha \tag{4}$$

where Γ denotes the Gamma function.

The rest of this paper is organized as follows: In Section 2, the description of the modified extended tanh-function method for solving nonlinear fractional partial differential equations is given. In Section 3, we apply this method to establish the exact solutions for the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations, the space–time fractional nonlinear Whitham–Broer– Kaup equations, the space–time fractional nonlinear coupled Burgers equations and the space–time fractional nonlinear coupled mKdV equations. In Section 4 physical explanations of some obtained solutions are given. In Section 5, some conclusions are obtained.

2. Description the modified extended tanh-function method for solving nonlinear fractional partial differential equations

Suppose we have the following nonlinear fractional partial differential equation:

$$F(u, D_t^\alpha u, D_x^\alpha u, \ldots) = 0, 0<\alpha\leqslant 1, \tag{5}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are the modified Riemann–Liouville derivatives and F is a polynomial in $u = u(x,t)$ and its fractional derivatives. In the following, we give the main steps of this method:

Step 1: Using the nonlinear fractional complex transformation (Li and He, 2010; Li, 2010; He and Li, 2012; Li et al., 2012; Hristov, 2010).

$$u(x,t) = u(\xi), \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} + \xi_0, \tag{6}$$

where k,c,ξ_0 are constants with k,$c \neq 0$, to reduce Eq. (5) to the following ODE of integer order with respect to the variable ξ :

$$P(u, u', u'', \ldots) = 0, \tag{7}$$

where P is a polynomial in $u(\xi)$ and its total derivatives u',u'',u''',... such that $u' = \frac{du}{d\xi}, u'' = \frac{d^2u}{d\xi^2}, \ldots..$

Step 2: We suppose that the formal solution of the ODE (7) can be expressed as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{N}[a_i\phi^i(\xi) + b_i\phi^{-i}(\xi)], \tag{8}$$

where $\phi(\xi)$ satisfies the Riccati equation

$$\phi' = b + \phi^2, \tag{9}$$

where b is a constant. Fortunately, Eq. (9) admits several types of the following solutions:

(i) If $b<0$, we have the hyperbolic solutions;

$$\phi(\xi) = -\sqrt{-b}\tanh(\sqrt{-b}\xi), \phi(\xi) = -\sqrt{-b}\coth(\sqrt{-b}\xi). \tag{10}$$

(ii) If $b>0$, we have the trigonometric solutions;

$$\phi(\xi) = \sqrt{b}\tan(\sqrt{b}\xi), \phi(\xi) = -\sqrt{b}\cot(\sqrt{b}\xi). \tag{11}$$

(iii) If $b=0$, we have the rational solutions;

$$\phi(\xi) = \frac{-1}{\xi+d}, \tag{12}$$

where d is a constant.

Step 3: We determine the positive integer N in (8) by balancing the highest nonlinear terms and the highest order derivatives of $u(\xi)$ in Eq. (7).

Step 4: We substitute (8) along with Eq. (9) into Eq. (7) and equate all the coefficients of $\phi^i(i=0, \pm 1, \pm 2,\ldots)$ to zero to yield a system of algebraic equations for a_i,b_i,c,k, b.

Step 5: We solve the algebraic equations obtained in Step 4 using Mathematica or Maple, and use the well- known solutions (10)–(12) of Eq. (9) to obtain the exact solutions of Eq. (5).

3. Applications

In this section, we construct the exact solutions of the following four nonlinear fractional partial differential equations using the proposed method of Section 2:

Example 1. The Space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations.

These equations are well-known (Guo et al., 2012; Zheng, 2012) and have the forms:

$$D_t^\alpha u - \frac{1}{2}D_x^{3\alpha}u + 3uD_x^\alpha u - 3D_x^\alpha(vw) = 0, \tag{13}$$

$$D_t^\alpha v + D_x^{3\alpha}v - 3uD_x^\alpha v = 0, \tag{14}$$

$$D_t^\alpha w + D_x^{3\alpha}w - 3uD_x^\alpha w = 0, \tag{15}$$

where $0<\alpha\leqslant 1$. Eqs. (13)–(15) can be used to describe the iteration of two long waves with different dispersion relations (Abazari and Abazari, 2012). When $\alpha = 1$, Eqs. (13)–(15) were first proposed in (Wu and Geng, 1999). When $0<\alpha\leqslant 1$, Eqs. (13)–(15) have been discussed in (Zheng, 2012) using the (G'/G)-expansion method and in (Guo et al, 2012) using the fractional sub-equation method. Let us now solve Eqs. (13)–(15) using the