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ORIGINAL ARTICLE

Eigenvalue approach to fractional order thermoelasticity for an infinite body with a spherical cavity

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Abstract In this article, we consider the problem of a thermoelastic infinite body with a spherical cavity in the context of the theory of fractional order thermoelasticity. The inner surface of the cavity is taken traction free and subjected to a thermal shock. The form of a vector–matrix differential equation has been considered for the governing equations in the Laplace transform domain. The analytical solutions are given by the eigenvalue approach. The graphical results indicate that the fractional parameter effect plays a significant role on all the physical quantities.

1. Introduction

Biot (1956) modified the classical uncoupled theory of thermoelasticity by eliminating the paradox that elastic changes have no effect on the temperature. The heat equations for both theories predict infinite speeds of propagation for heat waves. So, various generalized theories of thermoelasticity were developed. Lord and Shulman, 1967 established the first model generalized thermoelasticity theory (LS). Green and Lindsay (1972) proposed the temperature rate dependent thermoelasticity (GL) theory. During the second half of twentieth century, a large amount of work has been devoted to solving thermoelastic problems. This is due to their many applications in widely diverse fields. In the contexts of the thermoelasticity theories, the counterparts of our problem have been considered by using analytical and numerical methods (Abbas, 2008, 2012, 2014; Abbas and Abo-Dahab, 2014; Abbas and Kumar, 2014; Abbas and Othman, 2012; Abbas and Zenkour, 2013; Abd-alla and Abbas, 2002; Dhaliwal and Sherief, 1980; Sherief and Anwar, 1988, 1989; Sherief et al., 2004; Zenkour and Abbas, 2014a,b).

Fractional calculus has been used successfully to modify many existing models of physical processes e.g., viscoelasticity, chemistry, electronics, wave propagation and biology. One can state that the whole theory of fractional derivatives and integrals was established in the second half of the nineteenth century. Various definitions and approaches of fractional derivatives have become the main purpose of many studies. Youssef (2010) and Youssef and Al-Lehaibi (2010) established the fractional order generalized thermoelasticity of both weak and strong heat conductivity in the context of generalized

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thermoelasticity were considered, and the corresponding variational theorem for fractional order generalized thermoelasticity was developed. A new model of fractional heat equation established by Ezzat (2011b) and Ezzat and El-Karamany (2011a,b). In addition, Sherief et al. (2010) established a new model by using the form of the heat conduction law. Kumar et al. (2013) studied the plane deformation due to thermal source in fractional order thermoelastic media.

In this work, we consider fractional order generalized thermoelasticity of an infinite body with spherical cavity under thermal shock. The inversion of Laplace transform has been carried out numerically by applying a method of numerical inversion of Laplace transform based on Stehfest technique (Stehfest, 1970). Numerical results for physical quantities are represented graphically.

2. The Governing equation

The heat conduction equation takes the form, (El-Karamany and Ezzat, 2011; Ezzat, 2011a),

$$\left(K_{ij}T_{,j}\right)_{,i} = \left(\frac{\partial}{\partial t} + \frac{\tau_o^\alpha}{\Gamma(\alpha+1)}\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right)(\rho c_e T + \gamma T_0 e), \quad 0 < \alpha \le 1. \tag{1}$$

The equations of motion without body force take the form

$$\sigma_{ij,j} = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{2}$$

.

The constitutive equations are given by

$$\sigma_{ij} = 2\mu e_{ij} + [\lambda e - \gamma(T - T_0)]\delta_{ij}, \tag{3}$$

where T is the temperature; λ, μ are Lame's constants; T_0 is the reference temperature; α is the fractional parameter; c_e is the specific heat at constant strain; K_{ij} is the thermal conductivity; ρ is the density of the medium; τ_o is the thermal relaxation time; σ_{ij} are the components of stress tensor; t is the time; δ_{ij} is the Kronecker delta symbol; α_t is the coefficient of linear thermal expansion; u_i are the displacement vector components and e_{ij} are the components of strain tensor.

Now, we suppose elastic and homogenous infinite body with spherical cavity occupying the region $a \le r < \infty$. Because of the symmetry, all the state functions can be expressed in terms of the space variable r and the time variable t. In a spherical co-ordinate system (r, ϕ, ψ), the displacement components have the form

$$u_r = u(r,t), \quad u_\phi = u_\psi = 0. \tag{4}$$

The strain–displacement relations are

$$e_{rr} = \frac{\partial u}{\partial r},\ e_{\phi\phi} = \frac{u}{r},\ e_{\psi\psi} = \frac{u}{r},\ e_{r\phi} = e_{r\psi} = e_{\phi\psi} = 0, \tag{5}$$

$$e = \frac{\partial u}{\partial r} + 2\frac{u}{r}. \tag{6}$$

Thus, the stress–strain relations are

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda\left(\frac{\partial u}{\partial r} + 2\frac{u}{r}\right) - \gamma(T - T_0), \tag{7}$$

$$\sigma_{\phi\phi} = 2\mu\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + 2\frac{u}{r}\right) - \gamma(T - T_0), \tag{8}$$

$$\sigma_{\psi\psi} = 2\mu\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + 2\frac{u}{r}\right) - \gamma(T - T_0). \tag{9}$$

The equation of motion and energy equation have the form:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\phi\phi} - \sigma_{\psi\psi}) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{10}$$

$$K\frac{1}{r_2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \left(\frac{\partial}{\partial t} + \frac{\tau_o^\alpha}{\Gamma(\alpha+1)}\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right)\left(\rho c_e T + \gamma T_0\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right)\right) \tag{11}$$

For simplicity, we will use the following non-dimensional variables (Othman and Abbas, 2012).

$$(r', u') = \frac{(r,u)}{c\chi},\ (t', \tau_o') = \frac{(t, \tau_o)}{\chi},\ (\sigma_{rr}', \sigma_{\phi\phi}', \sigma_{\psi\psi}') = \frac{1}{\lambda + 2\mu}(\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{\psi\psi}),\ T' = \frac{\gamma(T - T_0)}{\lambda + 2\mu}, \tag{12}$$

where, $c^2 = \frac{\lambda+2\mu}{\rho},\ \chi = \frac{K}{\rho c_e c^2}$.

From Eq. (12) into Eqs. (7)–(11) one may obtain (after dropping the superscript $'$ for convenience)

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} - \frac{\partial T}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \tag{13}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \left(\frac{\partial}{\partial t} + \frac{\tau_o^\alpha}{\Gamma(\alpha+1)}\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right)\left(T + \varepsilon\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right)\right), \tag{14}$$

$$\sigma_{rr} = \frac{\partial u}{\partial r} + 2\beta\frac{u}{r} - T, \tag{15}$$

$$\sigma_{\phi\phi} = \sigma_{\psi\psi} = \beta\frac{\partial u}{\partial r} + (1 + \beta)\frac{u}{r} - T, \tag{16}$$

where $\beta = \frac{\lambda}{\lambda+2\mu},\ \varepsilon = \frac{\gamma T_0}{\rho^2 c^2 c_e}$.

From preceding description, we assume that the medium is initially at rest. The undisturbed state is maintained at reference temperature. Then we have

$$u(r,0) = \frac{\partial u(r,0)}{\partial t} = 0,\ T(r,0) = \frac{\partial T(r,0)}{\partial t} = 0. \tag{17}$$

The boundary conditions may be expressed as

$$\sigma_{rr}(a,t) = 0,\ T(a,t) = H(t), \quad \sigma_{rr}(r,t)|_{r\to\infty} = 0,\ T(r,t)|_{r\to\infty} = 0, \tag{18}$$

where $H(t)$ is the Heaviside unit step function.

3. Laplace Transform domain

Applying the Laplace transform define by the formula

$$\bar{f}(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt. \tag{19}$$

Hence, the Eqs. (13)–(18) take the form

$$\frac{d^2\bar{u}}{dr^2} + \frac{2}{r}\frac{d\bar{u}}{dr} - \frac{2\bar{u}}{r^2} - \frac{d\bar{T}}{dr} = s^2\bar{u}, \tag{20}$$

$$\frac{d^2\bar{T}}{dr^2} + \frac{2}{r}\frac{d\bar{T}}{dr} = \left(s + s^{1+\alpha}\frac{\tau_o^\alpha}{\Gamma(\alpha+1)}\right)\left(\bar{T} + \varepsilon\left(\frac{d\bar{u}}{dr} + \frac{2\bar{u}}{r}\right)\right), \tag{21}$$