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Terminal boundary-value technique for solving singularly perturbed delay differential equations

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Abstract

A terminal boundary-value technique is presented for solving singularly perturbed delay differential equations, the solutions of which exhibit layer behaviour. By introducing a terminal point, the original problem is divided into inner and outer region problems. An implicit terminal boundary condition at the terminal point was determined. The outer region problem with the implicit boundary condition was solved and produces an explicit boundary condition for the inner region problem. Then, the modified inner region problem (using the stretching transformation) is solved as a two-point boundary value problem. The second-order finite difference scheme was used to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. To validate the efficiency of the method, some model examples were solved. The stability and convergence of the scheme was also investigated.

MSC: 65L03; 65L10; 65L11; 65L12

Keywords: Singular perturbation; Delay differential equations; Finite differences; Terminal boundary condition; Boundary layer

1. Introduction

Singular perturbation problems containing a small parameter, ε, multiplying to their highest derivative term arise in many fields, such as fluid mechanics, fluid dynamics, chemical reactor theory and elasticity, which have received significant attention. The solution of these types of problems shows a multi-scale character, with a narrow region called the boundary layer, in which their solution changes rapidly, and an outer region in which the solution changes smoothly. Thus, the treatment of such problems is not trivial because of the boundary layer behaviour of their solutions. Detailed theory and analytical discussions of solving singular perturbation problems have been published [1–9], and have the details of numerical and asymptotic solutions [10–15].

Boundary value problems involving delay differential equations arise in studying the mathematical modelling of various practical phenomena, like micro-scale heat transfer [16], the hydrodynamics of liquid helium [17], second-sound theory [18], optically bi-stable devices [19], diffusion in polymers [20], reaction–diffusion equations [21], stability [22], control including control of chaotic systems [23] and a variety of models for physiological processes and diseases [24,25]. For example, Lange and Miura [26–30] treated the singular perturbation analysis of the problem under consideration in a series of papers. A numerical method based on the fitted mesh approach to approximate the solution

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of these types of boundary value problems was published by Kadalbajoo and Sharma [31]. The authors constructed piecewise-uniform meshes and fitted them to the boundary layer regions to adapt the singular behaviour of the operator in the narrow regions. The same authors [32] approximated the terms with delay by first-order Taylor series expansions to analyze boundary value problems of singularly perturbed differential difference equations with negative shift or delay. They used the invariant embedding technique and central and upwind finite difference discretization for the second- and first-order derivatives, respectively, and proved the stability and convergence of their method. An exponentially fitted difference scheme on a uniform mesh is accomplished by the method of integral identities to solve problems of the same type [33]. In this method, the authors used exponential basis functions and interpolating quadrature rules with weight and remainder terms in integral form. Various numerical methods have also been presented for solving singularly perturbed boundary value problems involving small shifts, including exponential methods based on piecewise analytical solutions of advection–reaction–diffusion operators [34], a fitted mesh B-spline collocation method [35], parameter–uniform numerical methods comprising a standard implicit finite difference scheme [36], ε-uniformly convergent non-standard finite differences [37] and ε-uniformly convergent fitted methods [38].

We devised a terminal boundary value technique for solving singularly perturbed delay differential equations with a boundary layer at the left end of the interval. By introducing a terminal point into the domain, the original problem is divided into inner and outer region problems. A terminal boundary condition in the implicit form is determined from the reduced problem, and the outer region problem with the implicit boundary condition is then solved as a two-point boundary-value problem. From the solution of the outer region problem, an explicit terminal boundary condition is obtained. The inner region problem is modified and solved as a two-point boundary value problem using the obtained explicit terminal boundary condition. Finally, we combined the solutions of the inner region and outer region problems to obtain the approximate solution of the original problem. The method is iterative on the terminal point. We repeated the numerical scheme for various choices of the terminal point until the solution profiles did not differ materially from iteration to iteration. To validate the efficiency of the method, some model examples are solved. The stability and convergence of the scheme was also investigated.

2. Description of the method

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{1}$$

Subject to the interval and boundary conditions

$$y(x) = \phi(x), \quad -\delta \le x \le 0 \tag{2}$$

$$y(1) = \beta, \tag{3}$$

where ε is a small positive parameter $(0<\varepsilon \ll 1)$, δ is delay parameter, $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are sufficiently smooth functions, and β is a known constant.

For $\delta = 0$, the problem (1)–(3) becomes a boundary value problem for singularly perturbed ordinary differential equations. The layer behaviour of the problem under consideration is maintained only for $\delta \neq 0$ but is sufficiently small (i.e. δ is of $o(\varepsilon)$). When the delay parameter δ exceeds the perturbation parameter, ε (i.e. δ is of $O(\varepsilon)$), the layer behaviour of the solution is no longer maintained; rather, the solution exhibits an oscillatory behaviour or is diminished.

We considered the cases in which δ is of $o(\varepsilon)$ $(i.e. \delta < \varepsilon)$. Now, we assume that $a(x) \ge M > 0$ and $\varepsilon - \delta a(x) > 0$ throughout the interval [0,1], where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$, which in general displays a boundary layer in the neighbourhood of $x = 0$.

Since δ is of $o(\varepsilon)$ and the solution $y(x)$ of the BVP (1)–(3) is sufficiently differentiable, by using Taylor's series expansion, we obtain

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{4}$$

Substituting (4) into (1), we obtain an asymptotically equivalent two-point boundary value problem

$$\varepsilon(1 - \xi a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{5}$$

with

$$y(0) = \phi(0) \tag{6}$$

$$y(1) = \beta \tag{7}$$

where $\delta = \xi\varepsilon$ with $\xi = O(1)$.

The transition from Eq. (1) to Eq. (5) is admitted because of the condition that the delay parameter $0 < \delta \ll 1$ is sufficiently small and is of $o(\varepsilon)$.

Let $x_p (0 < x_p \ll 1)$ be the terminal point or width or thickness of the boundary layer. It is well known from the singular perturbation theory (see [1,3]) that the