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On derivations and commutativity in prime near-rings

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Abstract

In the present paper it is shown that zero symmetric prime right near-rings satisfying certain identities are commutative rings.

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1. Introduction

In this paper N will denote a zero symmetric right near-ring (i.e., a right near ring N satisfying the property $x \cdot 0 = 0$ for all $x \in N$). Note that the right distributivity in N gives $0 \cdot x = 0$ for all $x \in N$. For any $x, y \in N$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. The symbol $Z(N)$ will represent the multiplicative center of N, that is, $Z(N) = \{x \in N \mid xy = yx$ for all $y \in N\}$. In the remainder part of this paper, unless otherwise specified, we will use the word near-ring to mean zero symmetric right near-ring and denote xy instead of $x \cdot y$. An additive mapping $d : N \to N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [16], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. According to [9], a near-ring N is said to be prime if $xNy = \{0\}$ for all $x, y \in N$ implies $x = 0$ or $y = 0$. Recently, there has been a great deal of work concerning commutativity of prime and semi-prime rings with derivations satisfying certain differential identities (see [2,8,10,15] for reference where further references can be found). In view of these results many authors have investigated commutativity of prime near-rings satisfying certain polynomial conditions (see [3–6,9–14,16], etc.). In the present paper it is shown that a near-rings with derivations satisfying certain identities are commutative rings.

2. Main results

We facilitate our discussion with the following lemmas which are required for developing the proofs of our main theorems. Note that the proofs of Lemmas 1, 3 and 4 can be seen in [7, Theorem 2.1], [4, Theorem 4.1] and [9, Lemma 3], respectively. Similar results can be obtained for right near-ring.

Lemma 1. *Let N be a prime near-ring. If N admits a nonzero derivation d for which $d(N) \subset Z(N)$, then N is a commutative ring.*

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Lemma 2. *Let d be an arbitrary derivation on the near-ring N. Then N satisfies the following partial distributive law:*

(i) $z(d(x)y+xd(y))=zd(x)y+zxd(y)$ *for all* $x, y\in N$.
(ii) $z(xd(y)+d(x)y)=zxd(y)+zd(x)y$ *for all* $x, y\in N$.

Proof. (*i*) By the simple calculation of $d(z(xy))=d((zx)y)$ we obtain the required result. Proof of (*ii*) can be seen in [1]. □

Lemma 3. *Let N be a 2-torsion free prime near-ring. If N admits a nonzero derivation d such that* $d([x, y])=0$ *for all* $x, y\in N$*, then N is a commutative ring.*

Lemma 4. *Let N be a 2-torsion free prime near-ring. If N admits a derivation d such that* $d^2=0$*, then* $d=0$.

Theorem 1. *Let N be a prime near-ring which admits a nonzero derivation d. Then the following assertions are equivalent*

(i) $d([x, y])=[d(x), y]$ *for all* $x, y\in N$.
(ii) $[d(x), y]=[x, y]$ *for all* $x, y\in N$.
(iii) *N is a commutative ring.*

Proof. It is easy to verify that (*iii*) ⇒ (*i*) and (*iii*) ⇒ (*ii*).

(*i*) ⇒ (*iii*) Assume that

$$d([x, y]) = [d(x), y] \quad \text{for all } x, y \in N. \tag{1}$$

Replacing y by yx in (1) we get

$$[d(x), yx] = d([x, y]x) \quad \text{for all } x, y \in N. \tag{2}$$

By definition of d, (2) reduced to

$$xyd(x) = yd(x)x \quad \text{for all } x, y \in N. \tag{3}$$

Substituting yz for y in (3) where $z\in N$, we obtain $[x, y]zd(x)=0$ which leads to

$$[x, y]Nd(x) = \{0\} \quad \text{for all } x, y \in N. \tag{4}$$

Since N is prime, Eq. (4) yields

$$d(x) = 0 \quad \text{or} \quad [x, y] = 0 \quad \text{for all } x, y \in N. \tag{5}$$

From (5) it follows that for each fixed $x\in N$ we have

$$d(x) = 0 \quad \text{or} \quad x \in Z(N). \tag{6}$$

But $x\in Z(N)$ also implies that $d(x)\in Z(N)$ and (6) forces $d(x)\in Z(N)$ for all $x\in N$, hence $d(N)\subset Z(N)$ and using Lemma 1, we conclude that N is a commutative ring.

(*ii*) ⇒ (*iii*) Suppose that

$$[d(x), y] = [x, y] \quad \text{for all } x, y \in N. \tag{7}$$

Replacing x by xy in (7), because of $[xy, y]=[x, y]y$, we get

$$[d(xy), y] = [x, y]y = [d(x), y]y \quad \text{for all } x, y \in N.$$

In view of Lemma 2(i), the last equation can be rewritten as

$$d(x)y^2 + xd(y)y - yxd(y) - yd(x)y = d(x)y^2 - yd(x)y,$$

so that

$$xd(y)y = yxd(y) \quad \text{for all } x, y \in N. \tag{8}$$

Since Eq. (8) is the same as Eq. (3), arguing as in the proof of (*i*) ⇒ (*iii*) we find that N is a commutative ring. □

Theorem 2. *Let N be a 2-torsion free prime near-ring which admits a nonzero derivation d. Then the following assertions are equivalent*

(i) $d([x, y])\in Z(N)$ *for all* $x, y\in N$.
(ii) *N is a commutative ring.*

Proof. It is clear that (*ii*) ⇒ (*i*).

(*i*) ⇒ (*ii*). We are given that

$$d([x, y]) \in Z(N) \quad \text{for all } x, y \in N. \tag{9}$$

(*a*) If $Z(N)=\{0\}$, it follows $d([x, y])=0$ for all $x, y\in N$. By Lemma 3, we conclude that N is a commutative ring.

(*b*) If $Z(N) \neq \{0\}$, replacing y by yz in (9), where $z\in Z(N)$, we get

$$d([x, y])z + [x, y]d(z) \in Z(N) \quad \text{for all } x, y \in N, z \in Z(N). \tag{10}$$

Using (9) together with Lemma 2(i), Eq. (10) implies

$$[x, y]d(z) \in Z(N) \quad \text{for all } x, y \in N, z \in Z(N).$$

Accordingly, $0=[[x, y]d(z), t]=[[x, y], t]d(z)$ for all $t\in N$ and thus

$$[[x, y], t]Nd(z) = \{0\} \quad \text{for all } x, y, t \in N, z \in Z(N). \tag{11}$$

Using the primeness of N, from (11) it follows that

$$d(Z(N)) = \{0\} \quad \text{or} \quad [[x, y], t] = 0 \quad \text{for all } x, y, t \in N.$$

Assume that $[[x, y], t]=0$ for all $x, y, t\in N$; substituting yx for y we get $[[x, y]x, t]=0$ and therefore $[x, y][x, t]=0$ for all $x, y, t\in N$. As $[x, y]\in Z(N)$, hence

$$[x, y]N[x, y] = \{0\} \quad \text{for all } x, y \in N. \tag{12}$$