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Journal of Taibah University for Science 8 (2014) 162–174

Journal of Taibah University for Science

www.elsevier.com/locate/jtusci

Complexity of some special named graphs with double edges

Salama N. Daoud [a,b,*], Kamel A. Mohamed [a,c]

[a] Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, El-Minufiya University, Shebeen El-Kom, Egypt
[c] Department of Sciences and Mathematics – New-Valley Faculty of Education, Assuit University, Egypt

Received 17 November 2012; accepted 30 August 2013
Available online 31 October 2013

Abstract

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we derive simple formulas of the complexity, number of spanning trees, of Some Special named Graphs with double edges such as Fan, Wheel and Mobius ladder, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

MSC: 05C05; 05C50

Keywords: Number of spanning trees; Fan; Wheel; Mobius ladders; Chebyshev polynomials

1. Introduction

In this introduction we give some basic definitions and lemmas. We deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G, also called, the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time) and appear in a number of applications. Most notable application fields are network reliability [1–3], enumerating certain chemical isomers [4] and counting the number of Eulerian circuits in a graph [5]. A classical result of Kirchhoff [6] can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \ldots, v_n\}$, then the Kirchhoff matrix H defined as $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H = [a_{ij}]$ defined as follows: (i) $a_{ij} = -1$ v_i and v_j are adjacent and $i \neq j$, (ii) a_{ij} equals the degree of vertex v_i if $i = j$, and (iii) $a_{ij} = 0$ otherwise. All of co-factors of H are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_1 \geq \ldots\ldots \geq \mu_p$ denote the

* Corresponding author at: Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, Saudi Arabia. Tel.: +966 598914649.
E-mail addresses: salamadaoud@gmail.com, sa_na_daoud@yahoo.com (S.N. Daoud), kamel16@yahoo.com (K.A. Mohamed).
Peer review under responsibility of Taibah University

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http://dx.doi.org/10.1016/j.jtusci.2013.08.002

eigenvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [7] shown that, $\tau(G) = (1/p)\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G) = (1/p)\prod_{k=1}^{p-1}(d-\mu_k)$ where $\lambda_0 = d, \lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such results is due to Cayley [8] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}$, $n\geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1}$, $p, q\geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [9,10]. Another result is due to Sedlacek [11] who derived a formula for the wheel on $n+1$ vertices, W_{n+1}, which is formed from a cycle C_n on n vertices by adding a vertex adjacent to every vertex of C_n. In particular, he showed that $\tau(W_{n+1}) = (3+\sqrt{5}/2)^n + (3-\sqrt{5}/2)^n - 2$, for $n\geq 3$. Sedlacek [12] also derived a formula for the number of spanning trees in a Mobius ladder. The Mobius ladder M_n is formed from cycle C_{2n} on $2n$ vertices labeled $v_1, v_2, \ldots, v_{2n}$ by adding edge $v_i v_{i+n}$ for every vertex v_i where $i \leq n$. The number of spanning trees in M_n is given by $\tau(M_n) = (n/2)[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n\geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch et al. [13,14]. Let the vertices of two disjoint and length cycles be labeled $v_1, v_2, \ldots, v_n$ in one cycle and $w_1, w_2, \ldots, w_n$ in the other. The prism R_n is defined as the graph obtained by adding to these two cycles all edges of the form $v_i w_i$. The number of spanning trees in R_n is given by the following formula $(n/2)[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2]$.

Later, Daoud [15–28] derived formulas for the number of spanning trees in Cocktail Party, Crown and Trapezoidal graphs.

2. Chebyshev polynomials

In this section we introduce some lemmas on determinants and some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, Yuanping et al. [29].

Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & 0 \\ -1 & 2x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2x \end{pmatrix},$$

where all other elements are zeros.

Further we recall that the Chebyshev polynomials of the first kind are defined by:

$$T_n(x) = \cos(n \arccos x) \tag{1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)} \tag{2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \tag{3}$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), \quad n\geq 1 \tag{4}$$