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Generalized class of fractional inverse Lagrange expansion

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Abstract

In this work the left and right Riemann–Liouville derivatives are introduced. A generalized operator based on left/right Riemann–Liouville derivatives is obtained. Basic definitions, lemmas and theorems in the fractional calculus which is related to our purpose are presented. The fractional form of Lagrange expansion theorem is obtained.

Keywords: Fractional right/left Riemann–Liouville derivative; Generalized operator; Lagrange expansion; Delta-function

1. Introduction

The development of the FC theory is due to the contributions of many mathematicians such as Euler, Liouville, Riemann, and Letnikov. Several definitions of a fractional derivative have been proposed. These definitions include Riemann–Liouville, Grunwald-Letnikov,Weyl, Caputo, Marchaud, and Riesz fractional derivatives, see [1–4]. Riemann–Liouville derivative is the most used generalization of the derivative. It is based on the direct generalization of Cauchy's formula for calculating an *n*-fold or repeated integral. The right and the left Riemann–Liouville fractional derivatives, in brief, are denoted by RRLFD and LRLFD respectively, see [5]. In [6–8] Agrawal has studied Fractional variational problems using the Riemann–Liouville derivatives. He notes that even if the initial functional problems only deal with the left Riemann–Liouville derivative, the right Riemann–Liouville derivative appears naturally during the computations. In what follows, we construct an operator combining the left and right Riemann–Liouville (RL) derivative. We remind some results concerning functional spaces associated to the left and right RL derivative. In particular, we discuss the possibility to obtain a law of exponents.

In 1770, Joseph Louis Lagrange (1736–1813) published his power series solution of the implicit equation. However, his solution used cumbersome series expansions of logarithms [9,10]. This expansion was generalized by Bürmann [11–13]. There is a straightforward derivation using complex analysis and contour integration; the complex formal power series version is clearly a consequence of knowing the formula for polynomials, so the theory of analytic functions may be applied. Actually, the machinery from analytic function theory enters only in a formal way in this proof. In

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1780, Pierre-Simon Laplace (1749–1827) published a simpler proof of the theorem, based on relations between partial derivatives with respect to the variable and the parameter, see [14,15]. Charles Hermite (1822–1901) presented the most straightforward proof of the theorem by using contour integration [16–18].

In mathematical analysis, this series expansion is known as Lagrange inversion theorem, also known as the Lagrange–Bürmann formula, and gives the Taylor series expansion of the inverse function. Suppose $z = f(w)$ where f is analytic function at a point a and $f(a) \neq 0$. Then it is possible to invert or solve the equation for w such that $w = g(z)$ on a neighborhood of $f(a)$, where g is analytic at the point $f(a)$. This is also called reversion of series. The series expansion of g is given by

$$w = g(z) = a + \sum_{n=1}^{\infty} \left(\lim_{w \to a} \left(\frac{(z - f(a))^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w - a}{f(w) - f(a)} \right)^n \right) \right)$$

In this work we will apply the concepts of fractional calculus to obtain a fractional form of Lagrange expansion and some generalizations.

Definition 1. By D, we denote the operator that maps a differentiable function onto its integer derivative, i.e. $Df(x) = f'$, by J_a, we denote the integer integration operator that maps a function f, assumed to be (Riemann) integrable on the compact interval [a,b], onto its primitive centered at a, i.e. $J_a\, f(x) = \int_a^x f(t)dt, \quad \forall \quad a \le x \le b.$

Definition 2. By D^n and J_a^n, $n \in \mathbb{N}$ we denote the n-fold iterates of D and J_a, respectively. Note that D^n is the left inverse of J_a^n in a suitable space of functions.

Lemma 1. *Let f be Riemann integrable on [a,b]. Then, for* $a \le x \le b$ *and* $n \in \mathbb{N}$*, we have*

$$J_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} f(t)dt, \quad n \in \mathbb{N}. \tag{1}$$

2. Left and right Riemann–Liouville derivatives and integrals

We define the left and right Riemann–Liouville derivatives following [19,20]. In all the following definitions $\Gamma(\alpha)$ represents the Euler gamma function given by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-\alpha} dx = (\alpha - 1)!$$

Definition 3. Let f be a function defined on $(a, b) \subset \mathbb{R}$, and $\alpha \in \mathbb{R}^+$. Then the left and right Riemann–Liouville fractional integral of order α is on respective defined on Lebesgue space $L_1[a, b]$ are given by

$${}_a\mathcal{D}_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - s)^{\alpha-1} f(s)ds, \quad a \le x \le b, \quad \alpha \in \mathbb{R} \tag{2}$$

$${}_x\mathcal{D}_b^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s - x)^{\alpha-1} f(s)ds, \quad a \le x \le b, \quad \alpha \in \mathbb{R} \tag{3}$$

Definition 4. Let f be a function defined on $(a, b) \subset \mathbb{R}$, and $\alpha \in \mathbb{R}^+$, the left and right Riemann–Liouville derivative of order α defined on Lebesgue space $L_1[a, b]$ and denoted by ${}_a\mathcal{D}_x^{-\alpha}$ and ${}_x\mathcal{D}_b^{\alpha}$ respectively, are defined by

$${}_a\mathcal{D}_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x - s)^{n-\alpha-1} f(s)ds \quad a \le x \le b, \quad \alpha \in \mathbb{R} \tag{4}$$

$${}_x\mathcal{D}_b^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_x^b (x - s)^{n-\alpha-1} f(s)ds, \quad a \le x \le b, \quad \alpha \in \mathbb{R} \tag{5}$$

Left and right (RL) integrals satisfy some important properties like the semi-group property.