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ORIGINAL ARTICLE

# On the nonlinear difference-differential equations arising in physics

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**Abstract** Here, an extended discrete tanh function method with a computerized symbolic computation is used constructing a new exact travelling wave solutions of nonlinear differential difference equations of special interest in physics, namely, Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations.

As a result, we obtain many kinds of exact solutions which include soliton solutions, periodic solutions and rational solutions in a uniform way if solutions of these kinds exist. The method is straightforward and concise, and it can also be applied to other nonlinear difference differential equations in physics.



## 1. Introduction

A large class of nonlinear evolution equations have been derived and widely applied in various branches of natural science. The investigation of the travelling wave solutions for nonlinear evolution equations arising in physics plays an important role in the study of nonlinear physical phenomena. A variety of powerful methods for obtaining the exact solutions of nonlinear evolution equations have been presented (Abdou and Soliman, 2005; Abdou, 2008, 2007, 2008a,b; Abdou and Zhang, 2009; Abulwafa et al., 2007, 2008; He and Abdou, 2007; He, 2006).

The nonlinear differential-difference equations (DDEs) have been the focus of many nonlinear studies. DDEs describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains (Ablowitz and Clarson, 1991; Kevrekidis et al., 2001; Tsuchida et al., 1999; Hirota, 2004; Qian et al., 2004; Ma and Geng, 2001) and so on.

DDEs play an important role in the study of modern physics and also play a crucial role in numerical simulations of nonlinear partial differential equations (NLPDEs), queueing problems, and discretization in solid state and quantum physics.

There have been developed many methods to solve DDEs, such as inverse scattering method (Tsuchida et al., 1999) and Hirota bilinear method (Hirota, 2004), Variables separate method (Qian et al., 2004), Buacklund transformation (Ma and Geng, 2001) and Darboux transformation can also be

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applied to solve DDEs and other methods (Xie and Wang, 2009; Dai and Zhang, 2006; Ahmet, 2009; Wang and Zhang, 2007; Baldwin et al., 2004; Wang, 2009).

The rest of the paper is organized as follows: In Section 2, extended discrete tanh-function method is presented. In Section 3, we choose three nonlinear difference differential equations, namely, Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations to illustrate the validity and advantage of this method. Finally conclusion and discussion are given in Section 4.

## 2. Methodology

In what follows is the summarized discrete tanh method (Wang, 2009). For a given nonlinear difference equations as

$$\psi_{n+1}=\frac{\psi_n+\epsilon A}{1+\delta A\psi_n}, \tag{1}$$

$$\psi_{n-1}=\frac{\psi_n-\epsilon A}{1-\delta A\psi_n^2},$$

$$\frac{\partial\psi_n}{\partial\xi_n}=\epsilon-\delta\psi_n^2, \tag{2}$$

where $\xi_n=kn+\lambda t$. For a given $\epsilon=1,\delta=1,\ A=tanh(k)$, the equation system has solutions $\psi_n=tanh(\xi_n)$ and $\psi_n=coth(\xi_n)$. In case of the parameters taken as $\epsilon=1,\delta=-1,A=tan(k)$, the equation system has solutions $\psi_n=tan(\xi_n)$ and $\psi_n=-cot(\xi_n)$. For the parameters taken as $\epsilon=0,\ \delta=1,A=k$, the equation system has solutions $\psi_n=\frac{1}{\xi_n}$.

For a given nonlinear difference differential equations as

$$\phi(u_{n+p_1}(x),u_{n+p_2}(x),u_{n+p_3}(x),\ldots,u_{n+p_k}(x),u'_{n+p_1}(x),$$
$$u'_{n+p_2}(x),..,u'_{n+p_k}(x),.$$

$$,u^r_{n+p_1}(x),u^r_{n+p_2}(x),\ldots,u^r_{n+p_k})=0, \tag{3}$$

where the dependent variable $u_n$ have $N$ components $u_{i,n}$ and so do its shifts; the continuous variable $x$ has $h$ components $x$, the discrete variable $n$ has $s$ components $n_j$; the $k$ shift vectors $p_i$, and $u^r(x)$ denotes the collection of mixed derivative terms of order $r$. By introducing wave transformation

$$u_n(x)=U(\xi_n)=U_n,\xi_n=\sum_{j=1}^{s}k_in_i+\sum_{j=1}^{h}\lambda_jx_j+c,$$

$$u_{n+p}(x)=U(\xi_{n+p})=U_{n+p},\xi_{n+p}=\sum_{j=1}^{s}k_i(n_i+p)+\sum_{j=1}^{h}\lambda_jx_j+c, \tag{4}$$

where, $k_i$ and $\lambda_j$ are all arbitrary constants to be determined later, $c$ is constant.

In the context of discrete tanh function method, many authors (Baldwin et al., 2004; Wang, 2009) used the ansatz

$$U(\xi_n)=\sum_{i=0}^{M}a_i\psi_n^i(\xi_n), \tag{5}$$

where $a_i$ are constants to be determined. In order to construct more general, it is reasonable to introduce the following ansatz

$$U(\xi_n)=\sum_{i=0}^{M}a_i\psi_n^i(\xi_n)+\sum_{j=1}^{M}b_j\psi_n^{-j}(\xi_n) \tag{6}$$

It is observed that the solution of the ansatz (8) goes back to that obtained from (7) once $b_j=0$, where $b_j$ are constants to be determined later, $\psi_n$ satisfy the system of Eqs. (1) and (2), $M$ can be determined.

Case(1): We set the degree of $U_{n+p}-U_{n-p},p\neq0$ is zero. For the other terms in DDE, $U_n$ is of degree $M_i$ is $\psi_n$ and $U_{n+p},\ p\neq0$ is of degree zero. Then we can balance the highest nonlinear terms and the highest linear terms to determine $M$ in Eq. (7). If $M$ is odd, then give the value of $M$ to $m_1$

Case(2): Replacing zero by $-1$ as the degree of $U_{n+p}-U_{n-p}$, $p\neq0$, we balance the term again. If $M$ is even, then give the value of $M$ to $m_2$, otherwise omit it.

The polynomial form expression of difference terms $U_{n+p}=\sum_{i=0}^{M}a_i\psi^i_{n+p}+\sum_{j=1}^{M}b_j\psi^{-j}_{n+p}$ in the similar way as Eq. (6). Inserting the expression of $U_n,\ldots\ldots,U_{n+p}$ in to Eq. (5) yields an ordinary DDE in terms of $\psi_n,\ldots,\psi_{n+p}$. With the aid of Eqs. (1)–(3), we reduce the equation obtained.

Collecting coefficients of all terms, $\psi_n^i, i=1,2,\ldots.$ and setting to zero yield a system of algebraic equation.

Solving the system of algebraic equations by computer algebra systems such as Maple.

Substituting the result relation obtained above and combining the solutions of Eqs. (1)–(3), we could get the solutions of given DDE Eq. (4)

## 3. New applications

To illustrate the effectiveness and the advantages of the proposed method, three models of nonlinear differential difference equations in physics are chosen, namely, Hybrid equation, Toda lattice equation and Relativistic Toda lattice difference equations. As a result, many exact travelling wave solutions are obtained including solitary wave solutions expressed by hyperbolic functions, periodic solutions expressed by trigonometric functions and rational solutions.

### 3.1. Example(1). Hybrid nonlinear difference differential equation

Let us first consider the Hybrid nonlinear differential difference equation (Baldwin et al., 2004),

$$\frac{\partial u_n}{\partial t}=\left(1+\alpha u_n+\beta u_n^2\right)(u_{n+1}-u_{n-1}), \tag{7}$$

where $\alpha$ and $\beta$ are constants. The Hybrid nonlinear difference Eq. (9) describes the discretization of the KdV and modified KdV equations. Making use the travelling wave solution as

$$u(n,t)=U(\xi_n),\quad \xi_n=kn+\lambda t+c, \tag{8}$$

where $k,\ c$ and $\lambda$ are constants. Then the discrete Hybrid nonlinear difference equation reduces to

$$\lambda\frac{\partial U_n}{\partial\xi_n}=\left(1+\alpha U_n(\xi_n)+\beta U_n^2(\xi_n)\right)(U_{n+1}(\xi_n)-U_{n-1}(\xi_n)) \tag{9}$$

Consider the balancing between the highest nonlinear term $\beta U_n^2(U_{n+1}-U_{n+1})$ with the highest derivative term $\lambda\frac{\partial U_n}{\partial\xi_n}$ according to case(2) mentioned above, we have $M=2$. We assume the solution of Eq. (11) can be expressed as

$$U_n(\xi_n)=a_0+a_1\psi_n(\xi_n)+a_2\psi_n^2(\xi_n)+\frac{b_1}{\psi_n(\xi_n)}+\frac{b_2}{\psi^2(\xi_n)}, \tag{10}$$

where $a_0,\ a_1,\ a_2,\ b_1$ and $b_2$ are to be determined later, $\psi_n$ satisfy the system of nonlinear Eqs. (1)–(3). With the aid of the