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ORIGINAL ARTICLE

A fractional model of Navier–Stokes equation arising in unsteady flow of a viscous fluid



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Abstract In this paper, we present a reliable algorithm based on the new homotopy perturbation transform method (HPTM) to solve a time-fractional Navier–Stokes equation in a tube. The fractional derivative is considered in the Caputo sense. By using an initial value, the explicit solution of the equation has been presented in a closed form and then its numerical solution has been represented graphically. The new homotopy perturbation transform method is a combined form of the Laplace transform method and the homotopy perturbation method. The results obtained by the proposed technique indicate that the approach is easy to implement and computationally very attractive.

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1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Fractional differential equations have gained importance and popularity, mainly due to their demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction–diffusion, turbulence, diffusive transport akin to diffusion, electric

networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and many other physical processes. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular (Caputo, 1969; Debnath, 2003; He, 1998, 1999a; Hilfer, 2000; Kilbas et al. 2006; Mainardi et al., 2001; Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999; Young, 1995).

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Recently, [El-Shahed and Salem \(2005\)](#) have generalized the classical Navier–Stokes equations by replacing the first time derivative by a fractional derivative of order α , $0 < \alpha \leq 1$. The Laplace, Fourier sine and finite Hankel transforms have been employed to obtain the exact solution for the time-fractional Navier–Stokes equations.

The Navier–Stokes equation is the primary equation of computational fluid dynamics, relating pressure and external forces acting on a fluid to the response of the fluid flow. The Navier–Stokes and continuity equations are given by:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u}, \quad (1)$$

$$\nabla \cdot \underline{u} = 0, \quad (2)$$

where ρ is the density, p is the pressure, ν is the kinematics viscosity, \underline{u} is the velocity and t is the time. This model can be generalized by replacing the first-time derivative by a fractional derivative of order α , $0 < \alpha \leq 1$. The time-fractional model for Navier–Stokes equation then has the form of the operator equation

$$D_t^\alpha \underline{u} + (\underline{u} \cdot \nabla) \underline{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{u}, \quad (3)$$

where D_t^α denotes the Caputo fractional derivative of order α . The time-fractional Navier–Stokes equations have been studied by [Ganji et al. \(2010\)](#); [Momani and Odibat \(2006\)](#) by using the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) respectively. The homotopy perturbation method (HPM) was first introduced by [He \(1999b\)](#). The homotopy perturbation method has also been used by many researchers to handle linear and nonlinear problems arising in science and engineering ([Ganji, 2006](#); [He, 2003, 2006](#); [Kumar and Singh, 2010](#); [Kumar, 2013](#); [Kumar et al., 2012a](#); [Vanani et al., 2013](#)). The homotopy analysis method was applied to study boundary layer flow in the region of the stagnation point towards a stretching sheet ([Nadeem et al., 2010a](#)) and stagnation flow of a Jeffrey fluid over a shrinking sheet ([Nadeem et al., 2010b](#)). In recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform. Among these are the Laplace decomposition method ([Gondal et al., 2013a](#); [Khuri, 2001](#); [Khan and Hussain, 2011](#); [Khan and Gondal, 2012a,b](#); [Khan et al., 2012a, 2012d, 2013](#); [Khan, 2013](#)), homotopy perturbation transform method ([Kumar et al., 2012b, 2012c](#); [Singh et al. 2012a, 2012b, 2013](#); [Khan et al., 2011, 2012b, 2012e](#); [Gondal and Khan, 2010](#)) and homotopy analysis transform method ([Arife et al., 2012](#); [Gondal et al., 2013b](#); [Khan et al. 2012c](#); [Kumar et al. 2013a, 2013b](#); [Khader et al., 2013](#); [Salah et al., 2013](#)).

In the present article, we consider the unsteady flow of a viscous fluid in a tube in which, besides time as one of the dependent variables, the velocity field is a function of only one space coordinate. Next, we apply the new homotopy perturbation transform method (HPTM) to solve the time-fractional Navier–Stokes equation. The homotopy perturbation transform method (HPTM) is a combination of the Laplace transform method and the homotopy perturbation method (HPM). The objective of this paper is to extend the application of the HPTM to obtain a solution of the time-fractional Navier–Stokes equation. The advantage of this technique is its capability of

combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

2. Basic definitions of fractional calculus

In this section, we mention the following basic definitions of fractional calculus.

Definition 1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu$, $\mu \geq -1$ is defined as ([Podlubny, 1999](#)):

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \quad (4)$$

$$J^0 f(t) = f(t). \quad (5)$$

For the Riemann–Liouville fractional integral we have:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \quad (6)$$

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense is defined as ([Caputo, 1969](#)):

$$D_t^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (7)$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$.

Definition 3. The Laplace transform of the Caputo derivative is given by Caputo ([Caputo, 1969](#)); see also [Kilbas et al. \(2006\)](#) in the form

$$L[D_t^\alpha f(t)] = s^\alpha L[f(t)] - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0+) \quad (n-1 < \alpha \leq n). \quad (8)$$

3. Analysis of the new proposed method

Consider unsteady, one-dimensional motion of a viscous fluid in a tube. The equations of motions which govern the flow field in the tube are the Navier–Stokes equations in cylindrical coordinates and they are given by.

$$\frac{\partial \underline{u}}{\partial t} = -\frac{\partial p}{\rho \partial z} + \nu \left(\frac{\partial^2 \underline{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \underline{u}}{\partial r} \right), \quad (9)$$

subject to the initial condition

$$u(r, 0) = f(r). \quad (10)$$

If the fractional derivative model is used to present the time derivative term, the Eq. (9) assumes the form

$$\frac{\partial^\alpha \underline{u}}{\partial t^\alpha} = P + \nu \left(\frac{\partial^2 \underline{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \underline{u}}{\partial r} \right) \quad 0 < \alpha \leq 1, \quad (11)$$

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