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ORIGINAL ARTICLE

# A new approach for a class of nonlinear boundary value problems with multiple solutions

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**Abstract** In this paper, an approach based on the variational iteration method (VIM) is proposed with an auxiliary parameter to predict the multiplicity of the solutions of homogeneous nonlinear ordinary differential equations with boundary conditions. The proposed approach is capable to predict and calculate all branches of the solutions simultaneously. Four practical problems are chosen to show the efficiency and importance of the proposed method. The proposed approach successfully detects multiple solutions to Bratu's problem, the model of mixed convection flows in a vertical channel, the nonlinear model of diffusion and reaction in porous catalysts and the nonlinear reactive transport model.



## 1. Introduction

Inokuti et al. (1978) proposed a general Lagrange multiplier method to solve nonlinear problems, which was intended to solve problems in quantum mechanics. Subsequently, He (1997) has modified the method to an iterative method and named it variational iteration method (VIM) and it has been presented by many authors to be a powerful mathematical tool for solving various types of nonlinear problems which represent plenty of modern science branches (He, 2012a, 2006; Yang and Baleanu, 2013; Wu, 2012). But this method cannot provide us with a simple way to adjust and control the convergence region and rate of giving approximate series, this reason was a strong motivation for authors to construct the variational iteration algorithms with an auxiliary parameter $h$ which proves very effective to control the convergence region of an approximate solution as (He, 2012b; Hosseini et al., 2011; Turkyilmazoglu, 2011; Ghaneai et al., 2012; Hosseini et al., 2010) and others.

It is very important to predict and calculate all solutions of nonlinear differential equations with boundary conditions in engineering and physical sciences. In this regard, many authors constructed the algorithms that are based on the homotopy analysis method (HAM) for multiple solution of nonlinear boundary value problems as Li and Liao, 2005; Abbasbandy and Shivanian, 2011; Abbasbandy et al., 2009; Hassan and El-Tawil, 2011; Hassan and Semary, 2013 and others. However, in this work, the algorithm based on the variational iteration method (VIM) with an auxiliary parameter is presented in prediction and actual determination of multiple solutions of nonlinear boundary value problems. To show the efficiency

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and importance of the proposed method, four practical problems are solved. A problem arising in mixed convection flows in a vertical channel (Barletta, 1999; Barletta et al., 2005), Bratu's problem (Wazwaz, 2012; Jalilian, 2010; Wazwaz, 2005), the nonlinear model of diffusion and reaction in porous catalysts (Sun et al., 2004; Abbasbandy, 2008; Magyari, 2008) and the nonlinear reactive transport model (Ellery and Simpson, 2011; Vosoughi et al., 2012), respectively and all of them admit multiple (dual) solutions which is why these models have been chosen to accomplish the article's goal.

## 2. Analysis of the method

Consider the nonlinear differential equation

$$Lu[(t)] + N[u(t)] = g(t), \tag{2.1}$$

where $L$ is a linear operator, $N$ is a nonlinear operator and $g(t)$ is an inhomogeneous term. According to the variational iteration method, one can construct an iteration formula for the Eq. (2.1) as follows:

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda(\tau)\{L[u_m(\tau)] + N[\tilde{u}_m(\tau)] - g(\tau)\}d\tau, \quad m \geqslant 0, \tag{2.2}$$

where $\lambda(\tau)$ is a general Lagrange multiplier, $\tilde{u}_m(\tau)$ is considered as a restricted variation (He, 1998; He, 1999; Wazwaz, 2007; Wazwaz, 2009) which means $\delta\tilde{u}_m(\tau) = 0$. To solve (2.1) by He's VIM (He, 1997), we first determine the Lagrange multiplier $\lambda(\tau)$ that can be identified optimally via variational theory. Then, the successive approximations $u_{m+1}(t)$, $m \geqslant 0$ of the solution $u(t)$ can be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(t)$. The initial approximation $u_0(t)$ may be selected by any function that just satisfies at least the initial and boundary conditions. With $\lambda(\tau)$ to be determined, several approximations $u_{m+1}(t)$, $m \geqslant 0$, follow immediately. Consequently, the exact solution may be obtained by using the form

$$u(t) = \lim_{m\to\infty} u_m(t). \tag{2.3}$$

Ghaneai et al. (2012) constructed a variational iteration algorithm with an auxiliary parameter in the form

$$u_{m+1}(t) = u_m(t) + h\int_0^t \lambda(\tau)\{L[u_m(\tau)] + N[\tilde{u}_m(\tau)] - g(\tau)\}d\tau, \quad m \geqslant 0, \tag{2.4}$$

where $h$ is an auxiliary parameter. The proposed approach to predict the multiplicity of the solutions of homogeneous nonlinear ordinary differential equations with boundary conditions based on the algorithm (2.4). Let the problem (2.1) be the form

$$\frac{d^s u(t)}{dt^s} + N[u(t)] = 0, \qquad s \geqslant 2 \tag{2.5}$$

with the boundary condition

$$\left.\frac{d^i u(t)}{dt^i}\right|_{t=0} = b_i, \quad i = 0, 1, \ldots, s-2, \quad u(a) = b. \tag{2.6}$$

Firstly by adding the new condition with unknown parameter $\alpha$ in the boundary conditions (2.6) and splitting into

$$\left.\frac{d^i u(t)}{dt^i}\right|_{t=0} = b_i, \left.\frac{d^{s-1} u(t)}{dt^{s-1}}\right|_{t=0} = \alpha, \tag{2.7}$$

$$u(a) = b, \tag{2.8}$$

where $u(a) = b$ is called the forcing condition that comes from original conditions (2.6). By calculating variation with respect to $u_m(\tau)$ for variational iteration formula (2.2) on the problem (2.5) with the new initial conditions (2.7), the Lagrange multiplier $\lambda(\tau)$ can be identified as (He, 1998; He, 1999; Wazwaz, 2007; Wazwaz, 2009)

$$\lambda(\tau) = \frac{-(t-\tau)^{s-1}}{(s-1)!}, \tag{2.9}$$

then the iteration formula (2.4) becomes

$$u_{m+1}(t,\alpha,h) = u_m(t,\alpha,h) - h\int_0^t \frac{(t-\tau)^{s-1}}{(s-1)!}\left\{\frac{d^s u_m(\tau,\alpha,h)}{d\tau^s} + N[u_m(\tau,\alpha,h)]\right\}d\tau. \tag{2.10}$$

It should be emphasized that $u_{m+1}(t,\alpha,h)$ can be computed by symbolic software programs such as Mathematica or Maple, starting by an initial approximation $u_0(t,\alpha)$ which satisfies at least the initial conditions (2.7). We obtain the approximate solution $u_{m+1}(t,\alpha,h)$ for the problem (2.5) and (2.7), but there are still two unknown parameters in the approximate solution $u_{m+1}$ $(t,\alpha,h)$ the unknown parameter $\alpha$ and the auxiliary parameter $h$, should be determined. The forcing condition (2.8) of the boundary value problem (2.5) reads

$$u(a) \approx u_{m+1}(a,\alpha,h) = b. \tag{2.11}$$

The Eq. (2.11) has two unknown parameters $\alpha$ and $h$ which control the approximation $u_{m+1}(t,\alpha,h)$ that converges to the exact solution. According to Eq. (2.11), $\alpha$ as function of $h$, by drawing the Eq. (2.11) gives the so called $h$-curve. The number of such horizontal plateaus where $\alpha(h)$ becomes constant, gives the multiplicity of the solutions. Also the horizontal plateaus indicate the convergence because the unknown parameter $\alpha$ is a constant value then a horizontal line segment in $h$-curve which corresponds to the valid region of $h$.

## 3. Applications

### 3.1. Bratu's problem

Consider Bratu's problem in one-dimensional planar:

$$\frac{d^2u}{dx^2} + \beta e^u = 0, \tag{3.12}$$

$$u(0) = u(1) = 0, \quad \beta > 0. \tag{3.13}$$

Bratu's problem (3.12) nonlinear two boundary value problem with strong nonlinear term $e^u$ and parameter $\beta$, appears in a number of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the Universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology (Wazwaz, 2012; Jalilian, 2010; Wazwaz, 2005). The analytical solution of (3.12) and (3.13) can be put in the following form: