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ORIGINAL ARTICLE

An extension of certain integral transform to a space of Boehmians

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Abstract This paper investigates the $\mathcal{L}^2$ transform on a certain space of generalized functions. Two spaces of Boehmians have been constructed. The transform $\mathcal{L}^2$ is extended and some of its properties are also obtained.

1. Introduction

Integral transforms are widely used to solve various problems in calculus, mechanics, mathematical physics, and some problems appear in computational mathematics as well.

In the sequence of these integral transforms, the Laplace - type integral transform, so-called $\mathcal{L}^2$ transform, is defined for a squared and an exponential function $f(t)$ by David et al. (2007), as

$$\mathcal{L}^2(f(x))(y) = \int_0^\infty x e^{-x^2y^2} f(x)dx. \tag{1}$$

The $\mathcal{L}^2$ transform is related to the classical Laplace transform by means of the following relationships :

$$\mathcal{L}^2(f(x))(y) = \frac{1}{2}\mathcal{L}\left(f\left(\sqrt{x}\right)\right)(y^2) \tag{2}$$

and

$$\mathcal{L}(f(x))(y) = 2\mathcal{L}^2(f(x^2))\left(\sqrt{y}\right). \tag{3}$$

Let f and g be Lebesgue integrable functions; then the operation $*$ between f and g is defined by

$$(f*g)(t) = \int_0^t xf\left(\sqrt{(t^2-x^2)}\right)g(x)dx.$$

The operation $*$ is commutative, associative and satisfies the equation $\mathcal{L}^2(f*g)(y) = \mathcal{L}^2(f)(y)\mathcal{L}^2(g)(y)$.

Some facts about the transform $\mathcal{L}^2$ are given as follows:

(1) $\mathcal{L}^2\left(\frac{\sin t^2}{t^2}\right)(y) = \frac{1}{2}\arctan\left(\frac{1}{y^2}\right)$.
(2) $\mathcal{L}^2(\mathcal{H}(t-a))(y) = \frac{1}{2y^2}e^{-y^2a^2}$, $\mathcal{H}$ being the heaviside unit function.
(3) $\mathcal{L}^2(t^2)(y) = \frac{\gamma\left(\frac{n}{2}+1\right)}{2y^{n+2}}$, γ being the gamma function.

More properties, applications and the inversion formula of $\mathcal{L}^2$ transform are given by Yürekli (1999a,b).

2. Abstract construction of Boehmians

The minimal structure necessary for the construction of Boehmians consists of the following elements :

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(*i*) A set $\Im$;
(*ii*) A commutative semigroup $(\Re, *)$;
(*iii*) An operation $\odot : \Im \times \Re \to \Im$ such that for each $x \in \Im$ and $\upsilon_1, \upsilon_2, \in \Re$,

$x \odot (\upsilon_1 * \upsilon_2) = (x \odot \upsilon_1) \odot \upsilon_2$;

(*vi*) A collection $\Delta \subset \Re^{\mathbb{N}}$ satisfying :
  (*a*) If $x, y \in \Im, (\upsilon_n) \in \Delta, x \odot \upsilon_n = y \odot \upsilon_n$ for all n, then $x = y$;
  (*b*) If $(\upsilon_n), (\sigma_n) \in \Delta$, then $(\upsilon_n * \sigma_n) \in \Delta$. Δ is the set of all delta sequences.

Consider

$\mathcal{A} = \{(x_n, \upsilon_n) : x_n \in \Im,\ (\upsilon_n) \in \Delta,\ x_n \odot \upsilon_m = x_m \odot \upsilon_n,\ \forall m,\ n \in \mathbb{N}\}$.

If $(x_n, \upsilon_n), (y_n, \sigma_n) \in \mathcal{A}, x_n \odot \sigma_m = y_m \odot \upsilon_n, \forall m, n \in \mathbb{N}$, then we say $(x_n, \upsilon_n) \sim (y_n, \sigma_n)$. The relation $\sim$ is an equivalence relation in $\mathcal{A}$. The space of equivalence classes in $\mathcal{A}$ is denoted by $\boldsymbol{b}(\Im, \Re, \Delta)$. Elements of $\boldsymbol{b}(\Im, \Re, \Delta)$ are called Boehmians.

Between $\Im$ and $\boldsymbol{b}(\Im, \Re, \Delta)$ there is a canonical embedding expressed as

$$x \to \frac{x \odot s_n}{s_n} \text{ as } n \to \infty.$$

The operation $\odot$ can be extended to $\boldsymbol{b}(\Im, \Re, \Delta) \odot \Im$ by

$$\frac{x_n}{\upsilon_n} \odot t = \frac{x_n \odot t}{\upsilon_n}.$$

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way

$$\left[\frac{(x_n)}{(\upsilon_n)}\right] + \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(x_n \odot \psi_n + g_n \odot \upsilon_n)}{(\upsilon_n \odot \psi_n)}\right]$$

and

$$\alpha\left[\frac{(x_n)}{(\upsilon_n)}\right] = \left[\frac{\alpha x_n}{\upsilon_n}\right],\ \alpha \in \mathbb{C}.$$

The operation $\odot$ and the differentiation are defined by

$$\left[\frac{(x_n)}{(\upsilon_n)}\right] \odot \left[\frac{(g_n)}{(\psi_n)}\right] = \left[\frac{(x_n \odot g_n)}{(\upsilon_n \odot \psi_n)}\right]$$

and

$$D^\alpha\left[\frac{(x_n)}{(\upsilon_n)}\right] = \left[\frac{(D^\alpha x_n)}{(\upsilon_n)}\right].$$

In particular, if $\left[\frac{(x_n)}{(\upsilon_n)}\right] \in \boldsymbol{b}(\Im, \Re, \Delta)$ and $\delta \in \Re$ is any fixed element, then the product $\odot$, defined by

$$\left[\frac{(x_n)}{(\upsilon_n)}\right] \odot \delta = \left[\frac{(x_n \odot \delta)}{(\upsilon_n)}\right],$$

is in $\boldsymbol{b}(\Im, \Re, \Delta)$.

Many a time $\Im$(also considered as a quasi-normed space) is also equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $\odot$ is given by:

(*i*) If $f_n \to f$ as $n \to \infty$ in $\Im$ and $\phi \in \Re$ is any fixed element, then $f_n \odot \phi \to f \odot \phi$ as $n \to \infty$ in $\Im$.
(*ii*) If $f_n \to f$ as $n \to \infty$ in $\Im$ and $(\delta_n) \in \Delta$, then $f_n \odot \delta_n \to f$ as $n \to \infty$ in $\Im$.

In $\boldsymbol{b}(\Im, \Re, \Delta)$, two types of convergence are:

(1) A sequence (h_n) in $\boldsymbol{b}(\Im, \Re, \Delta)$ is said to be δ-convergent to h in $\boldsymbol{b}(\Im, \Re, \Delta)$, denoted by $h_n \xrightarrow{\delta} h$ as $n \to \infty$, if there exists a delta sequence (υ_n) such that $(h_n \odot \upsilon_n), (h \odot \upsilon_n) \in \Im, \forall k, n \in \mathbb{N}$, and $(h_n \odot \upsilon_k) \to (h \odot \upsilon_k)$ as $n \to \infty$, in $\Im$, for every $k \in \mathbb{N}$.
(2) A sequence (h_n) in $\boldsymbol{b}(\Im, \Re, \Delta)$ is said to be Δ- convergent to h in $\boldsymbol{b}(\Im, \Re, \Delta)$, denoted by $h_n \xrightarrow{\Delta} h$ as $n \to \infty$, if there exists a $(\upsilon_n) \in \Delta$ such that $(h_n - h) \odot \upsilon_n \in \Im$, $\forall n \in \mathbb{N}$, and $(h_n - h) \odot \upsilon_n \to 0$ as $n \to \infty$ in $\Im$.

The following theorem is equivalent to the statement of δ- convergence :

Preposition 1. *$h_n \xrightarrow{\delta} h (n \to \infty)$ in $\boldsymbol{b}(\Im, \Re, \Delta)$ if and only if there is $f_{n,k}$, $f_k \in \Im$ and $\upsilon_k \in \Delta$ such that $h_n = \left[\frac{f_{n,k}}{\upsilon_k}\right]$, $h = \left[\frac{f_k}{\upsilon_k}\right]$ and for each $k \in \mathbb{N}$, $f_{n,k} \to f_k$ as $n \to \infty$ in $\Im$.*

For further discussion of Boehmian spaces and their construction; see Ganesan (2010), Karunakaran and Ganesan (2009), Al-Omari (2012, 2013a,b,c,d), Al-Omari and Kilicman (2011, 2012a,b, 2013a,b), Boehme (1973), Bhuvaneswari and Karunakaran (2010), Ganesan (2010), Karunakaran and Angeline (2011), Karunakaran and Devi (2010), Mikusinski (1983, 1987, 1995), Nemzer (2006, 2007, 2008, 2009, 2010) and Roopkumar (2009).

3. The Boehmian space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet,)$

$\boldsymbol{p}$ denotes the space of rapidly decreasing functions defined on $\mathbb{R}_+ (\mathbb{R}_+ = (0, \infty))$. That is, $\phi(x) \in \boldsymbol{p}$ if $\phi(x)$ is a complex-valued and infinitely smooth function defined on $\mathbb{R}_+$ and is such that, as $|t| \to \infty, \phi$ and its partial derivatives decrease to zero faster than every power of $|t|^{-1}$.

In more details, $\phi(t) \in \boldsymbol{p}$ iff it is infinitely smooth and is such that

$$|t^m \phi^{(k)}(t)| \leqslant C_{m,k},\ \ t \in \mathbb{R}_+, \tag{4}$$

m and k run through all non-negative integers; see Pathak (1997).

$\boldsymbol{d}$ denotes the Schwartz space of test functions of bounded support defined on $\mathbb{R}_+$.

$\bullet$ denotes the Mellin-type convolution product **of**first kind defined by Zemanian (1987), as

$$(f \bullet g)(y) = \int_0^\infty f\left(\frac{y}{t}\right) t^{-1} g(t) dt. \tag{5}$$

To construct the first Boehmian space $\boldsymbol{b}(\boldsymbol{p}, (\boldsymbol{d}, \bullet), \bullet, \Delta)$, we need to establish the following necessary theorems.

Theorem 2. *Let $\phi \in \boldsymbol{p}$ and $\varphi \in \boldsymbol{d}$; then we have $\phi \bullet \varphi \in \boldsymbol{p}$.*

Proof. Let $\mathbb{K}$ be a compact subset in $\mathbb{R}_+$containing the support of φ. Then, for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$, we, by (4) and (5), get that

$$\left|y^m \mathcal{D}_y^k (\phi \bullet \varphi)(y)\right| \leqslant \int_{\mathbb{K}} \left|y^m \mathcal{D}_y^k \left(\phi\left(\frac{y}{t}\right) t^{-1} \varphi(t)\right)\right| dt$$

$$= \int_{\mathbb{K}} |y^m| \left|\mathcal{D}_y^k \phi\left(\frac{y}{t}\right)\right| |t^{-1} \varphi(t)| dt \leqslant C_{m,k} \int_{\mathbb{K}} |t^{-1} \varphi(t)| dt < MC_{m,k}.$$