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# On adequate rings

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**Abstract**

In this paper, we investigate the transfer of notion of adequate rings to trivial ring extensions and pullbacks. Our aim is to give new classes of commutative rings satisfying this property.


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*Keywords:* Adequate ring; Trivial rings extension; Pullbacks

## 1. Introduction

All rings in this paper are commutative with unity. We denote by $U(R)$ the set of unit of a ring $R$. And, if $a, b \in R$, $a|b$ means $a$ divides $b$, that is $b = ac$ for some $c \in R$.

We know that an elementary divisor ring is a Hermite ring. Kaplansky showed that for the class of adequate domains being a Hermite ring was equivalent to being an elementary divisor ring. Gillman and Henriksen showed that this was also true for rings with zero-divisors. See for instance [4,7,10,12].

Now, we give the definition of adequare ring.

**Definition 1.1.** A ring $A$ is said an adequate ring if for all $a \in A - \{0\}$ and $b \in A$, there exists two non-zero elements $r$, $s$ of $A$ such that:

(a) $a = rs$.
(b) $rA + bA = A$.
(c) $\forall t \in A - U(A)$: $t$ divides $s$ implies $tA + bA \neq A$.

The notion of an adequate domain was originally defined by Helmer [7]. The ring of entire functions on the complex plane is an adequate Bezout domain (see [7,12]).

Let $A$ be a ring and $E$ an $A$-module. The trivial ring extension of $A$ by $E$ (also called idealization of $E$ over $A$) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$.

Trivial ring extensions have been studied extensively. Considerable work, part of is summarized in Glaz's book [5] and Huckaba's book [8], has been concerned with trivial ring extension. These extensions have been useful for solving many open problems and conjectures in both

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commutative and non-commutative ring theory. See for instance [1,5,8,9].

Let $T$ be a domain and let $K$ be a field which is a retract of $T$, that is $T := K + M$ where $M$ is a maximal ideal of $T$. Each subring $D$ of $K$ determines a subring $R := D + M$ of $T$. This construction arises frequently in algebra, especially in connection with counterexamples. The original of $D + M$ construction involved a valuation domain $T$ with $K := T/M$, where $M$ is the maximal ideal of $T$ and $K \subset T$. A throughout account of results about $D + M$ construction can be find in [2,3,5].

In this paper, we investigate the transfer of the adequate notion to trivial ring extensions and pullbacks. Our results generate new families of adequate rings.

## 2. Transfer of adequate property to trivial ring extension

We begin by showing that a local ring is adequate.

**Theorem 2.1.** *Let $(A, M)$ be a local ring. Then $A$ is an adequate ring.*

**Proof.** Let $(A, M)$ be a local ring, $a \in A - \{0\}$ and let $b \in A$. Two cases are then possible:

**Case 1**: $b \in M$.

It suffices to take $r = 1$ and $s = a$. Indeed:

(a) $a = 1a = rs$.
(b) $rA + bA = 1A + bA = A$.
(c) $\forall t \in A - U(A)$: $t$ divides $s$. Since $t \notin U(A)$, then $t \in M$. So, $tA + bA \subset M + M = M \neq A$. Therefore, $tA + bA \neq A$.

**Case 2**: $b \notin M$.

Then $b \in U(A)$ and so $bA = A$. Hence, it suffices to take $r = a$ and $s = 1$. Indeed:

(a) $a = a1 = rs$.
(b) $rA + bA = rA + A$.
(c) For each $t \in A - U(A)$, $t$ divides $s$. Using the fact $t \notin U(A)$ and $t$ divides 1, then $tA + bA \neq A$, as desired.

Hence, $A$ is an adequate ring.□

Now, we study the transfer of the adequate property to trivial ring extension.

**Theorem 2.2.** *Let $A$ be an integral domain, $E$ be an $A$-module such that $aE = E$ for each $a \in A - \{0\}$, and let $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then $R$ is an adequate ring if and only if the following conditions hold:*

(1) *$A$ is an adequate ring.*
(2) *$\forall a, b \in A - U(A)$, $aA + bA \neq A$.*

The proof of this theorem requires the following lemmas.

**Lemma 2.3.** *Let $A$ be a ring, $E$ an $A$-module, $R := A \propto E$ be the trivial ring extension of $A$ by $E$, and let $\pi$ be a projection of $R$ to $A$. Then:*

(1) *Let $a, b, c \in A$ and let $x, y, z \in E$. Then, $(a, x) = (b, y)(c, z)$ implies $a = bc$.*
(2) *Let $a, b \in A$ and $x, y \in E$. If $(b, y)|(a, x)$, then $b|a$.*
(3) *Let $a, b, c \in A$ such that $bE = E$ and let $x, y \in E$. The following statements are equivalents:*
   (i) *There exists $z \in E$ such that $(a, x) = (b, y)(c, z)$.*
   (ii) *$a = bc$.*
(4) *Let $a, b \in A$ such that $bE = E$ and let $x, y \in E$. Then, $(b, y)|(a, x)$ if and only if $b|a$.*
(5) *Assume that $A$ is an integral domain, $K := qf(A)$ be the quotient field of $A$ and $E$ be a $K$-vector space. Let $a, b, c \in A$ such that $b \neq 0$ and let $x, y \in E$. Then the following statement is equivalents:*
   (i) *There exists $z \in E$ such that $(a, x) = (b, y)(c, z)$.*
   (ii) *$a = bc$.*
(6) *Assume that $A$ is an integral domain, $K := qf(A)$ be the quotients fields of $A$ and let $E$ be a $K$-vector space. Let $a \in A$, $b \in A - \{0\}$, and let $x, y \in E$. Then, $(b, y)|(a, x)$ if and only if $b|a$.*
(7) *Let $I$ be an ideal of $A$ and $F$ a submodule of $E$ such that $IE \subset F$. Then, for each $t \in A$, we have:*

$$(t, 0)(I \propto F) = (tI \propto tF).$$

*In particular, if $I$ is an ideal of $A$, then for eacht $\in A$, we have $(t, 0)(I \propto E) = (tI) \propto (tE)$.*
(8) *Let $I$ be an ideal of $A$, $F$ be a submodule of $E$ such that $IE \subset F$, and let $(t, u) \in R := A \propto E$. Then, $\pi((t, u)(I \propto E)) = tI$.*
   *In particular, if $I$ is an ideal of $A$ and $(t, u) \in R$, then $\pi((t, u)(I \propto E)) = tI$.*
(9) *Let $a, b \in A$, and $x, y \in E$. Then, $(a, x)R + (b, y)R = R$ if and only if $aA + bA = A$.*

**Proof.** Straightforward. □

**Lemma 2.4.** *Let $R := A \propto E$ be the trivial ring extension of $A$ by the $A$-module $E$. Assume that $aE = E$ for each $a \in A - \{0\}$. If $R$ is an adequate ring, then so is $A$.*

**Proof.** Assume that $R$ is an adequate ring and $aE = E$ for each $a \in A - \{0\}$.