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Review Article

# Global analysis of a model of bioreactor with cell recycling using general form of rate functions

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**Abstract**

A dynamical model of an activated sludge process is considered. A mathematical analysis of the model equations with general rate functions is given. The dissipativity, boundedness, invariance of non-negativity, persistence, stability, bifurcation and periodicity of solutions are discussed. It was also shown that the rate functions have no influence for stability in the special case disinfected feed.


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*Keywords:* Nonlinear systems; Activated sludge process; Boundedness; Stability; Bifurcations

**Contents**



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## 1. Introduction

There has been considerable interest in the dynamics and control of activated sludge processes (see [1–4]). In [5], the authors discussed the effects of substrate inhibited kinetics in activated sludge reactors. It was shown that substrate inhibition models are fundamental in predicting the results of wastewater. The authors in ([6,7]) investigated the stability and the control of steady states multiplicity in the continuous stirred tank bioreactor (CSTBR). In [6], the authors discussed the stability in the CSTBR with cell recycle using the singularity theory.

In this paper we consider the following model of an activated sludge process proposed as Substrate ($S$) → Particulate product ($X_s$) → Biomass ($X_a$). It is known (see [1]) that the model is structured upon the following two processes:

(i) Formation of an intermediate particulate product ($X_s$) depending on substrate.
(ii) Active biomass ($X_a$) synthesis.

This leads to the following system

$$\begin{aligned} \theta_1 \frac{ds}{dt} &= S_f - S - \theta_1 X_a U_1(S, X_s), \\ \theta_1 \frac{dX_s}{dt} &= X_{s_f} - \omega X_s + \theta_1 X_a (U_1 - U_2), \\ \theta_1 \frac{dX_a}{dt} &= X_{a_f} - \omega X_a + \theta_1 X_a U_2(X_s), \end{aligned} \tag{1}$$

where $S$ is the substrate concentration, $X_s$ intermediate particulate product concentration, $X_a$ biomass concentration, $f$ (subscript) feed stream, $\theta_1$ reactor residence time and $\omega$ sludge withdrawal fraction. Also, the parameters $\theta_1$, $S_f$ and $\omega$ are positive and $X_{s_f}$, $X_{a_f}$ be non-negative.

The aim of this paper is to study the dynamic characteristics of a flexible model of activated sludge process with solid recycle with rate functions more general than those given by [1,8–10]. We discuss some qualitative properties of solutions such as boundedness, existence, dissipativity, positively invariance, persistence and stability, using techniques depending on those given by [11–13]. We also discuss the existence of Hopf bifurcation and periodic solutions of the system. In Section 2, we start to simplify the model and discuss some qualitative properties of equilibria such as boundedness, invariance, persistence, bifurcation and stability for the special case $\left(X_{s_f} = X_{a_f}\right)$, with a technique similar to that given by [12]. Section 3 is devoted for the general case of the model. In Sections 4 and 5 we discuss Hopf bifurcation and periodic process for the general form of the model. A numerical results and conclusions are given in Section 6.

## 2. Preliminaries

In this section we state assumptions and substitutions to simplify the system (1), then we start to study some qualitative properties of a special case of the model.

Now setting $S = s$, $X_s = x$, $X_a = z$, $1/\theta_1 = \theta$ then system (1) becomes:

$$\begin{aligned} \frac{ds}{dt} &= \theta S_f - \theta s - z U_1(s, x), \\ \frac{dx}{dt} &= \theta X_{s_f} - \theta \omega x + z(U_1 - U_2), \\ \frac{dz}{dt} &= \theta X_{a_f} - \theta \omega z + z U_2 \end{aligned} \tag{2}$$

where $s(t_0) = s_0 > 0$, $x(t_0) = x_0 > 0$, $z(t_0) = z_0 > 0$, $0 \le s < S_f$, $0 \le x < X_{s_f}$, $0 \le z < X_{a_f}$.

Suppose the following hypotheses

(**H₁**) $U_1$ has the properties