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# A study of near-rings with generalized derivations

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**Abstract**

In the present paper it is shown that 3-prime left near-rings satisfying certain identities involving generalized derivations are commutative rings. Moreover, examples proving the necessity of the 3-primeness hypothesis in various theorems are given.
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## 1. Definitions and terminology

Throughout this paper $N$ will be a zero-symmetric left near-ring with multiplicative center $Z(N)$; and usually $N$ will be 3-prime, that is, will have the property that $xNy = 0$ for $x, y \in N$ implies $x = 0$ or $y = 0$. For any $x, y \in N$, as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie and Jordan products respectively. Recalling that $N$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in N$. An additive mapping $d : N \longrightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$, or equivalently, as noted in [1], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. Many results in literature indicate how the global structure of a near-ring $N$ is often tightly connected to the behavior of additive mappings defined on $N$. More recently several authors consider similar situation in the case the derivation $d$ is replaced by a generalized derivation. According to [2], an additive mapping $F : N \longrightarrow N$ is said to be a right (resp., left) generalized derivation with associated derivation $d$ if $F(xy) = F(x)y + xd(y)$ (resp., $F(xy) = d(x)y + xF(y)$), for all $x, y \in N$, and $F$ is said to be a generalized derivation with associated derivation $d$ on $F$ if it is both a right and a left generalized derivation on $N$ with associated derivation $d$ (note that this definition differs from the one given by Hvala in [3]; his generalized derivations are our right generalized derivations.) Every derivation on $N$ is a generalized derivation. Familiar examples of generalized derivations are derivations and generalized inner derivations and later includes left multiplier i.e. an additive mapping $F : N \longrightarrow N$ satisfying $F(xy) = F(x)y$ for all $x, y \in N$. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting. Recently, there has been a great deal of work concerning commutativity of prime and semiprime rings

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admitting suitably constrained derivations and generalized derivations (see [4–7]). In view of these results many authors have proved comparable results for near-rings (see [8–11]). In the present paper it is shown that near-rings with generalized derivations satisfying certain identities are commutative rings. Many of our results extend earlier commutativity results involving similar conditions on derivations.

The following Lemmas are essential for developing the proofs of our results.

**Lemma 1.** *Let $N$ be a 3-prime near-ring.*

(i) [2, Lemma 1.2 (iii)] *If $z \in Z(N) \setminus \{0\}$ and $x \in N$ such that $xz \in Z(N)$, then $x \in Z(N)$.*
(ii) [1, Lemma 2] *If $d$ is a derivation of $N$ and $x \in Z(N)$, then $d(x) \in Z(N)$.*

**Lemma 2.** [4, Lemma 3 (ii), (iii) and (iv)] *Let $N$ be a 3-prime near-ring and $d$ is a derivation.*

(i) *If $Z(N) \setminus \{0\}$ contains an element $z$ for which $z + z \in Z(N)$, then $(N, +)$ is abelian.*
(ii) *If $x \in N$ such that $d(N)x = \{0\}$ or $xd(N) = \{0\}$, then $x = 0$ or $d = 0$.*
(iii) *If $N$ is 2-torsion free and $d$ is a derivation on $N$ such that $d^2 = 0$, then $d = 0$.*

**Lemma 3.** [2, Theorem 2.1] *Let $N$ be a 3-prime near-ring. If $N$ admits a derivation $d$ such that $d(N) \subseteq Z(N)$, then $N$ is a commutative ring.*

**Lemma 4.** [2, Lemma 1.1] *If $N$ is an arbitrary left near-ring and $d$ is a derivation, then $(d(x)y + xd(y))z = d(x)yz + xd(y)z$ for all $x, y, z \in N$.*

**Lemma 5.** [3, Lemma 5] *Let $N$ be a 3-prime near-ring. If $F$ is generalized derivation associated with a derivation $d$, then $(d(x)y + xF(y))z = d(x)yz + xF(y)z$ for all $x, y, z \in N$.*

## 2. Conditions involving right generalized derivations

**Theorem 1.** *Let $N$ be a 2-torsion free 3-prime near-ring. If $F$ is a nonzero right generalized derivation on $N$, then the following statements are equivalent:*

(i) *$d(Z(N)) \neq \{0\}$ and $F([x, y]) \in Z(N)$ for all $x, y \in N$;*
(ii) *$[F(x), y] \in Z(N)$ for all $x, y \in N$;*
(iii) *$F(x) \circ y \in Z(N)$ for all $x, y \in N$;*
(iv) *$N$ is a commutative ring.*

**Proof.** It is obvious that $(iv)$ implies $(i)$, $(ii)$ and $(iii)$.

$(i) \Rightarrow (iv)$ Let us fix $z \in Z(N)$ such that $d(z) \neq 0$. We are given that

$$F([x, y]) \in Z(N) \quad \text{for all} \quad x, y \in N. \tag{1}$$

Replacing $y$ by $yz$ in (1), we get

$$F([x, y])z + [x, y]d(z) \in Z(N) \quad \text{for all} \quad x, y \in N \tag{2}$$

which implies that

$$[x, y]d(z) \in Z(N). \tag{3}$$

Since $d(z) \in Z(N) - \{0\}$, then by view of Lemma 1 (i), (3) implies that

$$[x, y] \in Z(N) \quad \text{for all} \quad x, y \in N \tag{4}$$

and therefore

$$[[x, y], t] = 0 \quad \text{for all} \quad t, x, y \in N.$$

Substituting $xy$ for $y$, we get $[x[x, y], t] = 0$ which, because of $[x, y] \in Z(N)$, yields $[x, y][x, t] = 0$ for all $t, x, y \in N$. Accordingly,

$$[x, y]N[x, y] = 0 \quad \text{for all} \quad x, y \in N. \tag{5}$$

Once again using the 3-primeness, Eq. (5) forces $x \in Z(N)$ and therefore

$$d(x) \in Z(N) \quad \text{for all} \quad x \in N. \tag{6}$$

Hence $d(N) \subset Z(N)$ and Lemma 3 assures that $N$ is a commutative ring.

$(ii) \Rightarrow (iv)$ If $Z(N) = \{0\}$, then our hypothesis becomes $F(N) \subset Z(N)$ and [5, Theorem 2.1] assures that $N$ is a commutative ring. Hence, we assume that $Z(N) \neq \{0\}$. From

$$[F(x), y] \in Z(N) \quad \text{for all} \quad x, y \in N$$

it follows that

$$[[F(x), y], t] = 0 \quad \text{for all} \quad t, x, y \in N. \tag{7}$$

Replacing $y$ by $F(x)y$ in equation (7), we get

$$[[F(x), y]F(x), t] = 0 \quad \text{for all} \quad t, x, y \in N$$

and therefore $[F(x), y][F(x), t] = 0$ for all $t, x, y \in N$. Accordingly,

$$[F(x), y]N[F(x), y] = 0 \quad \text{for all} \quad x, y \in N. \tag{8}$$

Using the 3-primeness, equation (8) leads to

$$[F(x), y] = 0 \quad \text{for all} \quad x, y \in N.$$

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