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Ternary reversible number-conserving cellular automata are trivial

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ABSTRACT

We introduce a novel method to study the reversibility of d-dimensional number-conserving multi-state cellular automata with the von Neumann neighborhood. We apply this method to ternary such cellular automata, for which, up to now, nothing was known about their reversibility. It turns out that they are all trivial: the only reversible such cellular automata are shifts that are intrinsically 1-dimensional.

1. Introduction

The reversibility of cellular automata (CAs) has received significant attention since John von Neumann and Stanisıaw Ulam introduced these discrete dynamical systems in the late 1940s. This is due to the fact that many researchers have further developed the theory of CAs to use them as a suitable computational tool for the simulation of physical systems. Thus, as reversibility is one of the fundamental laws of physics at the microscopic scale, especially in quantum mechanics, CAs should preferabley exhibit the same property.

The first studies on the reversibility of CAs are due to Hedlund [19] and Richardson [29]. In particular, they proved that a CA is reversible if and only if it is a injection. In the literature one can now find a number of results on this topic. Among them are results regarding infinite CAs as well as finite CAs with various boundary conditions. However, the studies on the reversibility of CAs concentrate on 1-dimensional ones and linear CAs with $\mathbb{Z}_m$ as state set.

When it comes to 1-dimensional CAs, it is safe to say that the problem has been completely resolved. Indeed, an effective way to determine reversibility of infinite 1-dimensional CAs was shown by Amoroso and Patt [1] and also by Di Gregorio and Trautteur [12] and Sutner [33]. For finite CAs, an algorithm to decide the reversibility was developed by Bhattacharjee and Das [3]. Moreover, there are many results concerning specific Elementary CAs (ECAs) (see, for example, [8,9,16,30]). However, it is worth emphasizing that existing tools do not allow to enumerate all reversible 1-dimensional CAs even in the case of few states (of course, except in the case of ECAs), because it is impossible to check all CAs with a given state set, due to the inhibitive cardinality of the search space.

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While 1-dimensional reversible CAs have been studied in detail and from many different points of view, the same cannot be said about 2- or higher-dimensional ones. This is a consequence of the negative result of Kari [22] stating that there is no algorithm that can decide whether or not an arbitrary 2-dimensional CA is reversible.

Another class of CAs (1- or higher-dimensional) for which researchers have found methods to determine reversibility, is the class of linear CAs (LCAs). The reason that LCAs have been extensively studied, especially with $\mathbb{Z}_m$ as state set, is that they are particularly amenable to theoretical analysis. Firstly, the linearity of local rules opens the door to the realm of linear algebra and one can use powerful tools like transition matrices, since an LCA is reversible if and only if its transition matrix is reversible (see, for example, [7,10,26,32]). Secondly, the fact that the state set is a finite commutative ring allows to use formal power series representations to state necessary and sufficient conditions for the reversibility of an LCA in terms of the coefficients of its local rule. This method has been initiated by Itô et al. [21] and intensively developed by other researchers (see, for example, [11,25,37]). Thus, also in the case of LCAs with the state set $\mathbb{Z}_m$, we can say that the problem of reversibility has been solved completely. In particular, LCAs have been thoroughly examined in terms of ergodic theory: it was shown that a reversible LCA is either a Bernoulli automorphism or non-ergodic [6]. More information about the reversibility problem of these types of CAs can be found in [23,24,27].

However, the transition matrix tool has fundamental limitations, since the matrix size depends on the number of cells, so, it is useless when we deal with infinite CAs and for finite CAs, every grid size has to be considered separately. Moreover, applying matrix theory for a ring $\mathbb{Z}_m$, which does not have to be a field, one has to be very careful (see [2]). Also, the linearity assumption as well as the use of a finite ring as state set are often not acceptable. For example, in particle motion modeling, one would like two plus one to be three instead of zero as in the case of the field $\mathbb{Z}_3$.

The aim of this paper is to present a method that allows to study the reversibility of another type of CAs: number-conserving cellular automata (NCCAs). NCCAs represent a particularly interesting class of CAs that have the special feature of preserving the sum of the states upon every update of the states (see, for example, [13]). As state set, we will consider $\{0, 1, \ldots, q_*\}$, where q_* is a given natural number. CAs with such a state set will be referred to as multi-state CAs. As we will deal with d-dimensional CAs, we have to define the cell neighborhood. We opt for the closed unit ball in the d-dimensional Manhattan distance metric, *i.e.*, the von Neumann neighborhood, the most popular neighborhood used in modeling physical phenomena (for example, it is "the most common in overland flow models" [5]). Unfortunately, studying multidimensional CAs with this kind of neighborhood is very complicated, because the von Neumann neighborhood is not a Cartesian product of one-dimensional neighborhoods (in contrast to the Moore neighborhood).

So far, there are only a few results concerning the reversibility of multi-state NCCAs. Schranko and De Oliveira [31] performed numerical experiments involving many rules to finally conjecture that the class of 1-dimensional such CAs is too restrictive to be computationally universal. This was proved in the case of radius 1/2 CAs [18], but it does not hold in general as shown by Morita [28]. Boccara and Fukś [4] found all 144 1-dimensional ternary NCCAs, but only three among them are reversible: the identity rule and two shift rules. Imai et al. [20] listed all reversible 1-dimensional NCCAs with radius 1 and up to four states through an exhaustive search and they also constructed some with five states. In the case of two dimensions, the situation does not look better. Dzedzej et al. [15], using a characterization of number-conserving local rules given in [35], found that for the state set {0, 1} or {0, 1, 2}, *i.e.*, when we deal with binary or ternary 2-dimensional NCCAs with the von Neumann neighborhood, there are only trivial reversible ones, namely, the identity and the shifts (in each of the four possible directions). However, recently all four-state reversible 2-dimensional NCCAs with the von Neumann neighborhood have been enumerated [14], which until then was considered an intricate task because of two main reasons. Firstly, as we mentioned earlier, there is no algorithm to decide whether or not a given 2-dimensional CA is reversible. Secondly, there were no tools available to enumerate all four-state 2-dimensional NCCAs, so it was not possible to use the method of exhaustive search. The method used in [14] is based on the split-and-perturb decomposition of the local rule of an NCCA with the von Neumann neighborhood, proved in [36]. The same novel approach to study number conservation of CAs was used to find all binary d-dimensional NCCAs with the von Neumann neighborhood in [34], proving that for any given d there are exactly $4d+1$ such cellular automata: the identity rule and the shift and traffic rules in each of the $2d$ possible directions. Thus, if we ask for reversible binary NCCAs in d dimensions, then only the identity rule and the shift rules remain, since the traffic rules are not reversible.

In this paper, we want to combine the ideas presented in [14] and [34] to create a method that allows to enumerate all reversible ternary d-dimensional NCCAs with the von Neumann neighborhood. It is known that the behavior of a CA can be quite different depending on whether we are considering finite, periodic or infinite configurations (see, for example, [23]). In our investigations, we adhere to the weakest version of CA reversibility: we consider a given CA on a fixed finite d-dimensional grid (with periodic boundary conditions) and we require its global function to be an injection. We show that there are only trivial reversible ternary NCCAs, namely, the identity and the shifts (in each of $2d$ possible directions). The proof of this fact is given in the case of a finite grid. However, our results also valid in the case of periodic and infinite CAs, as we discuss at the end of Section 3.

This paper is organized as follows. Section 2 contains preliminaries and a description of the main idea of the split-and-perturb decomposition of a number-conserving local rule. The formulation and the proof of the main theorem are given in Section 3. Conclusions, conjectures and open problems are discussed in Section 4.

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