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A metric space of subcopulas — An approach via Hausdorff distance ☆

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Abstract

In this work, we define a distance function on the set of bivariate subcopulas to generate a compact metric space. Moreover, the copula space equipped with the uniform distance is essentially a metric subspace of this subcopula space. We also characterize the convergence in this space, and provide the interrelationship with the convergence of the distribution functions.

Keywords: Copula; Subcopula; Hausdorff metric; Subcopula convergence

1. Introduction

A *copula* is a joint distribution function of random variables uniformly distributed on the unit interval $\mathbb{I} = [0, 1]$. A subcopula can be considered as a copula restricted to a closed subset $\mathbb{A} \times \mathbb{B}$ of $\mathbb{I}^2$ containing $\{0, 1\}^2$. Thus, a subcopula can always be extended to a copula. However, copula extensions of a subcopula are usually not unique, unless, for example, that subcopula is a copula itself (see also Proposition 3.4 and Proposition 3.3). Amo et al. [3] provide a complete characterization of the copula extensions of any subcopula (see also Theorem 2.2).

Let $\mathcal{S}$ denote the set of all subcopulas. We are interested in defining a topology on $\mathcal{S}$. This topology should be an extension of a topology on the space of copulas $\mathcal{C}$, so that $\mathcal{C}$ could be viewed as a subspace of $\mathcal{S}$. One difficulty in defining such a topology arises because the domains of the subcopulas may vary. For example, the expression $C(x) - D(x)$ might be irrelevant unless $x \in \operatorname{dom}(C) \cap \operatorname{dom}(D)$. Therefore, it is not possible to define a distance function on $\mathcal{S}$ either as expressed in Equation (2.3) or Equation (2.4).

In this work, we overcome the above-mentioned difficulty by identifying a subcopula with a set of copulas extending it. We then define the distance between subcopulas as the Hausdorff distance between the sets of copulas extending them. This yields a topology on $\mathcal{S}$ with good properties. Specifically, the space $\mathcal{S}$ is a compact metric space, and the

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convergences under this topology can be characterized by conditions that do not require any information on the extensions.

According to Sklar's Theorem [12], there is a one–one correspondence between the set of joint distribution functions H with fixed marginals F and G and the set of subcopulas S with domain $\overline{\text{Range}(F)} \times \overline{\text{Range}(G)}$, such that $H(x, y) = S(F(x), G(y))$ for all $x, y \in \mathbb{R}$. It is also interesting to understand the relation between the convergence of the subcopulas and that of the distribution functions, particularly in the case where the marginal distributions are not fixed. We show in this work that the convergence of the joint distribution functions is equivalent to the convergence of their marginals and the associated subcopulas. Consequently, we find that empirical subcopulas always converge to the true subcopulas. The latter can then be used to study the dependence structure between non-continuous random variables analogous to the use of the convergence of empirical copulas to examine the dependence structure between continuous random variables.

This paper is organized as follows. In the next section, we discuss the basic terminologies and concepts used throughout this work. In particular, the Hausdorff distance and copula extensions of a subcopula. In Section 3, we describe the basic properties of the classes of copula extensions. In Section 4, we define a distance on the space of subcopulas and characterize the convergence of a sequence in this space. In the last section, we compare the convergences in the subcopula space and the distribution space.

2. Preliminaries

This section is divided into two parts. The first part provides the background material on the Hausdorff metric. The second part presents the related material on copulas and subcopulas available in the literature.

Let (X, d) be a compact metric space. Let $K(X)$ denote the set of all non-empty closed subsets of X. The *Hausdorff metric* of (X, d) is the function $h_d : K(X) \times K(X) \to [0, \infty)$ defined by

$$h_d(A, B) = \max\left(\max_{x\in A}\min_{y\in B} d(x, y), \max_{y\in B}\min_{x\in A} d(x, y)\right) \tag{2.1}$$

for all $A, B \in K(X)$. Theorem C.9 in [9] implies that $(K(X), h_d)$ is boundedly compact, i.e., any closed bounded subset of $K(X)$ is compact. Since (X, d) is compact, h_d must be bounded, and hence, $(K(X), h_d)$ is actually a compact metric space. The fact that (X, d) is compact also implies the equivalence of the convergences in $(K(X), h_d)$ and the convergence in the view of the Painleve–Kuratowski definition [9, Definition B.4 and Definition B.5]. This is proved by the immediate application of Corollary C.6 and Theorem C.2 (iii) in [9]. This fact will be used throughout this work; therefore, it is restated in the following theorem.

Theorem 2.1. *Let (X, d) be a compact metric space. The sequence K_n converges to K in $(K(X), h_d)$ if and only if K_n converges to K in the view of the Painleve–Kuratowski definition, i.e., if and only if the following conditions hold:*

1. *For any $x_{n(k)} \in K_{n(k)}$ such that $n(k) \to \infty$ and $x_{n(k)} \to x$ as $k \to \infty$, x must belong to K.*
2. *For any $x \in K$, there is $x_n \in K_n$ such that $x_n \to x$.*

The above theorem also implies that the topology of $(K(X), h_d)$ depends only on the topology of (X, d) but not on the metric d itself. In addition, $h_d(\{x\}, \{y\}) = d(x, y)$ for all $x, y \in X$. Therefore, the metric space (X, d) is embedded in $(K(X), h_d)$ as a subspace. For example, $h_d(x, K) = h_d(\{x\}, K)$. For further information on the Hausdorff metric, refer to the appendices B and C of [9].

Recall that a *subcopula* is a function $S : \mathbb{A} \times \mathbb{B} \to \mathbb{I}$, where $\mathbb{A}$ and $\mathbb{B}$ are closed subsets of $\mathbb{I}$ containing zero and one, satisfying the following conditions:

(G) $S(a, 0) = 0 = S(0, b)$ for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$,
(U) $S(a, 1) = a$ and $S(1, b) = b$ for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$, and
(2I) $V_S([a, b] \times [c, d]) \geq 0$, where

$$V_S([a, b] \times [c, d]) = S(b, d) - S(a, d) - S(b, c) + S(a, c) \tag{2.2}$$

for all $a, c \in \mathbb{A}$ and $b, d \in \mathbb{B}$ such that $a \leq c$ and $b \leq d$.

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