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Contractive sequences in fuzzy metric spaces

Valentín Gregori [a,*], Juan-José Miñana [b], David Miravet [a]

[a] *Universitat Politècnica de València, Instituto de Investigación para la Gestión Integrada de Zonas Costeras, Campus de Gandia, C. Paranimf 1, 46730, Spain*

[b] *Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears, Carretera de Valldemossa km. 7.5, 07122 Palma, Spain*

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Abstract

In this paper we present an example of a fuzzy ψ-contractive sequence in the sense of D. Mihet, which is not Cauchy in a fuzzy metric space in the sense of George and Veeramani. To overcome this drawback we introduce and study a concept of strictly fuzzy contractive sequence. Then, we also make an appropriate correction to Lemma 3.2 of Gregori and Miñana (2016) [5].

Keywords: Fuzzy metric space; Fuzzy contractive mapping; Fixed point

1. Introduction

In this paper we deal with the concept of fuzzy metric due to George and Veeramani [1] which is a modification of the one given by Kramosil and Michalek [9,3]. If M is a fuzzy metric on X then M generates a topology τ_M on X in a similar way to classical metrics. This topology is metrizable [2,7] and consequently, topics related to metrics have been systematically extended and studied in this fuzzy setting. In particular, (fuzzy) fixed point theory is a field of high activity.

Recall that in classical fixed point theory, a contractive sequence of iterates $\{f^{(n)}(x_0)\}$ of a self-contractive mapping f on a complete metric space X is constructed, for all $x_0 \in X$. This sequence converges in X since contractive sequences are Cauchy. But, what about this statement in fuzzy setting? We notice that in [8] the authors introduced a concept of fuzzy contractive sequence and they posed the following question: Is every fuzzy contractive sequence a Cauchy sequence? So far, there is no answer to this question (D. Mihet [11] gave a negative answer, but for fuzzy metrics in the sense of Kramosil and Michalek). The purpose of this article is to make a new contribution to this field and, at the same time, to correct an error appeared in [5]. For it, we will introduce and study a concept of *strictly fuzzy contractive sequence*.

Regarding the last paragraph, on the one hand, we notice that there are several concepts of Cauchy sequence in the literature [6]. Here we focus our attention in the two concepts used in fuzzy fixed point theory. The first one was given

* Corresponding author.
E-mail addresses: vgregori@mat.upv.es (V. Gregori), jj.minana@uib.es (J.-J. Miñana), damifor@alumni.uv.es (D. Miravet).

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by M. Grabiec in [3] and it will be denoted by G-Cauchy (see Definition 2.2). The second one will be called, simply, Cauchy (see Definition 2.3) and it is due to George and Veeramani [1] (although it comes from PM-spaces [13]). It is well known that Cauchy implies G-Cauchy. On the other hand, with respect to (fuzzy) contractive mappings, we deal with four related concepts, named within brackets, due (chronologically) to Gregori and Sapena (GS) [8], Mihet (ψ-contractive) [11], Romaguera and Tirado (RT) [12], and Wardowski ($\mathcal{H}$) [15] (see Definition 2.4). The relationship among these concepts is shown in the following chain of (strict) implications:

$$\text{RT-contractive} \implies \text{GS-contractive} \implies \mathcal{H}\text{-contractive} \implies \psi\text{-contractive}$$

According to these concepts, and in a similar way to classical metrics, we obtain their corresponding concepts of (fuzzy) contractive sequence (see Definition 2.5), which preserve the aforementioned chain of implications. In this paper we study which of these contractive conditions implies Cauchyness.

We observe in Proposition 3.11 that every ψ-contractive sequence is G-Cauchy. Nevertheless, our main result is Example 3.12 in which we construct a ψ-contractive sequence in a stationary fuzzy metric, which is not Cauchy. This example points out that the concept of ψ-contractivity needs to be strengthened, to some *strictly fuzzy contractivity*, to get Cauchyness. But in what form should it be done? Our decision is based on (the proof of) Lemma 3.2 of [5], which asserts: "A ψ-contractive sequence $\{x_n\}$ satisfying $\bigwedge_{t>0} M(x_1, x_2, t) > 0$, is a Cauchy sequence". It is clear, at the light of Example 3.12, that this lemma is false. The error in the mentioned proof is due to the fact that the authors have improperly used the property: $M(x_{m+1}, x_{n+1}, t) \geq \psi(M(x_m, x_n, t))$, for all $m, n \in \mathbb{N}$. Therefore we just define a strictly fuzzy ψ-contractive sequence (see Definition 3.2) as a ψ-contractive sequence that satisfies that property. In this manner, the mentioned Lemma 3.2 is valid for strictly fuzzy ψ-contractive sequences satisfying $\bigwedge_{t>0} M(x_1, x_2, t) > 0$.

In a similar way, the other three concepts of strictly fuzzy contractive sequence are defined, and among these four concepts, again the above chain of implications is satisfied. The given concept of strictly fuzzy contractive sequence can be considered an appropriate concept, not only because it makes true Lemma 3.2 aforementioned, but also because for $x_0 \in X$ the sequence of iterates $\left\{f^{(n)}(x_0)\right\}$ of a contractive mapping f, for each one of the mentioned contractive conditions, is strictly fuzzy contractive (see Proposition 3.5). The reader can find another favorable argument to this new concept in Proposition 3.7. Moreover, in (*b*) of Example 3.9 we give a convergent RT-contractive sequence, which is not strictly RT-contractive.

After properly correcting Corollary 3.8 and Lemma 3.12 of [5] we show two large classes of fuzzy metric spaces where the condition of strictly fuzzy contractivity for a sequence implies Cauchy.

The structure of the paper is as follows. Section 2 is dedicated to preliminaries. Section 3 contains the concept of strictly fuzzy ψ-contractive sequence and related results. Section 4 is, basically, a correction to [5].

2. Preliminaries

Definition 2.1 *(George and Veeramani [1]).* A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t norm and M is a fuzzy set on $X \times X \times]0, +\infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M_{xy} :]0, +\infty[\to]0, 1]$ is continuous, where $M_{xy}(t) = M(x, y, t)$.

The continuous t-norms commonly used in fuzzy logic are the minimum ($\wedge$), the usual product ($\cdot$), and the Lukasiewicz t-norm ($\mathfrak{L}$).

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$, or simply M, is a *fuzzy metric* on X. This terminology will also be extended along the paper in other concepts, as usual, without explicit mention.

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in]0, 1[$ and $t > 0$.

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