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Two Problems on Interval Counting

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Abstract

Let $\mathcal{F}$ be a family of intervals on the real line. An interval graph is the intersection graph of $\mathcal{F}$. An interval order is a partial order $(\mathcal{F}, \prec)$ such that for all $I_1, I_2 \in \mathcal{F}$, $I_1 \prec I_2$ if and only if I_1 lies entirely at the left of I_2. Such a family $\mathcal{F}$ is called a model of the graph (order). The interval count of a given graph (resp. order) is the smallest number of interval lengths needed in any model of this graph (resp. order). The first problem we consider is related to the classes of graphs and orders which can be represented with two interval lengths, regarding to the inclusion hierarchy among such classes. The second problem is an extremal problem which consists of determining the smallest graph or order which has interval count at least k. In particular, we study a conjecture by Fishburn on this extremal problem, verifying its validity when such a conjecture is constrained to the classes of trivially perfect orders and split orders.

Keywords: Extremal problems. Interval count. Interval graphs. Interval orders. Split graphs. Trivially perfect graphs.

1 Introduction

In graph theory, a well known graph class is that of interval graphs, which is strictly related to interval orders, also a well known topic in order theory. Interval graphs initially appeared in the areas of genetics and combinatorics [1]. A graph is an *interval graph* if its vertex set is a family of intervals on the real line, called a *model*, in which two distinct intervals are adjacent in the graph if such intervals intersect. An *interval order* is a partial order on a family of intervals on the real line in which the precedence relation correspond that of the intervals, that is, if the interval I_a precedes the interval I_b in the order, then I_a is entirely at the left of I_b.

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Ronald Graham raised the question of how many distinct interval lengths are required to any model of a given interval graph (cf. [2,8]). In other words, he suggested the problem of determining a model of a given interval graph having the smallest number of distinct interval lengths, which is called the *interval count* problem. A survey on the interval count parameter appeared in [1]. The problem of deciding efficiently whether a graph or order admits a model using a unique length is solved, since it is equivalent to the problem of recognizing unit interval graphs and orders (see [1]). However, deciding efficiently whether a graph or order admits a model using at most two lengths is open. Graham conjectured that, for every graph G, $IC(G) \leq k+1$ if $IC(G \backslash \{u\}) = k$ for some vertex $u \in V(G)$. Leibowitz, Assmann and Peck [2] proved it for $k = 1$ and provided a counterexample to such a conjecture for all $k \geq 2$, by presenting a graph G for each $k \geq 2$ such that, for a special vertex $u \in V(G)$, $IC(G \backslash \{u\}) = k$ and $IC(G) = k + 2$. Trotter [3] conjectured that the removal of a vertex can arbitrarily decrease the interval count (by a non-constant value), this is still open.

Although the study of the interval count parameter dates back the seventies, and despite all the efforts, most of the suggested problems of that time remain open until now. Klavík, in his excellent recent thesis on the subject, wrote:

> *The classes* k-LengthINT [graphs having interval count k] *were introduced by Graham as a natural hierarchy between unit interval graphs and interval graphs (...). Even after decades of research, the only results known are curiosities that illustrate the incredibly complex structure of such a class, very different from the case of unit interval graphs.* ([9], page 36)

In this work, we present new results on two problems. First, we study subclasses of graphs and orders having interval count at most two. An $\{a, b\}$-*model* is a model in which each interval has length a or b. The class of graphs which admit an $\{a, b\}$-model is denoted by $\textsc{Len}(a, b)$. Skrien [4] provided a characterization for $\textsc{Len}(0, 1)$. Rautenbach and Szwarcfiter [5] described a characterization and a linear-time algorithm to recognize graphs of $\textsc{Len}(0, 1)$. Boyadzhiyska, Isaak and Trenk [6] presented a characterization both for interval orders which admit a $\{0, 1\}$-model and for those which admit a $\{1, 2\}$-model. In [6], the question regarding the inclusion relation among these two classes was not considered. That motivated us to study the inclusion relation among the classes $\textsc{Len}(a', b')$ and $\textsc{Len}(a, b)$ for all $0 \leq a' < b'$, $0 \leq a < b$. Our results are described in Section 3. Except for the trivial case when $\textsc{Len}(a', b') = \textsc{Len}(a, b)$ which occurs when the lengths are proportional ($\frac{a'}{b'} = \frac{a}{b}$), $\textsc{Len}(a', b')$ surprisingly neither contains nor is contained in $\textsc{Len}(a, b)$.

Regarding the interval count parameter, more generally, it is not known how to efficiently decide whether the interval count of a graph or order is at most k, for all $k \geq 2$ (see [1]). Fishburn [7] studied the problem of determining the smallest order (in number of elements) and graph (in number of vertices) which requires k or more distinct interval lengths for their models. More specifically, this problem consists of determining the value of the function $\sigma_{\mathcal{C}}(k)$ (resp. $\tilde{\sigma}_{\mathcal{C}}(k)$) defined as the smallest number of intervals for which there is an order (resp. a graph) belonging to class $\mathcal{C}$ which has interval count at least k. Fishburn [7] formulated a conjecture to the