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On total f-domination: Polyhedral and algorithmic results

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ABSTRACT

Given a graph $G = (V, E)$ and integer values f_v, $v \in V$, a node subset $D \subset V$ is a total f-dominating set if every node $v \in V$ is adjacent to at least f_v nodes of D. Given a weight system $c(v)$, $v \in V$, the minimum weight total f-dominating set problem is to find a total f-dominating set of minimum total weight. In this article, we propose a polyhedral study of the associated polytope together with a complete and compact description of the polytope for totally unimodular graphs and cycles. We also propose a linear time dynamic programming algorithm for the case of trees.

1. Introduction

Let $G = (V, E)$ denote a simple graph having node set $V = [\![1, n]\!]$ and edge set E, where $[\![1, n]\!]$ stands for the set of integers $\{1, 2, \ldots, n\}$. For each $v \in V$, let d_v denote its degree in G and let f_v be a given nonnegative integer value. Let $\mathcal{F}_G$ stand for the set of vectors $\{f \in \mathbb{Z}^n_+ : 0 \le f_v \le d_v, \forall v \in V\}$. A node subset $D \subseteq V$ is called an *f-dominating set* (resp. a *total f-dominating set*) if each node $v \in V \setminus D$ (resp. $v \in V$) has at least f_v neighbor(s) in D. In the special case $f_v = 1$, for all $v \in V$, node set D is called a *dominating set* (resp. a *total dominating set*), see [13,14,17]. We consider then the *minimum weight total f-dominating set problem*, denoted by $[MWT_f]$: Given a simple graph $G = (V, E)$ with node weights $c_v \in \mathbb{R}$, for all $v \in V$, and $f \in \mathcal{F}_G$, find a minimum weight total f-dominating set of G, i.e. find a node subset $D \subseteq V$ such that D is a total f-dominating set and the weight of D: $\sum_{v \in D} c_v$, is minimum. This problem may be formulated as the integer program

$$(IP) \quad \left\{ \min \sum_{v \in V} c_v x_v : \sum_{u \in N(v)} x_u \ge f_v, \forall v \in V;\ x \in \{0, 1\}^n \right\},$$

where $N(v) = \{u : [u, v] \in E\}$ denotes the neighboring nodes of v. Its linear relaxation (obtained replacing the constraints $x \in \{0, 1\}^n$ by $x \in [0, 1]^n$) will be denoted by (P). Given a node subset $S \subseteq V$, let $\chi^S \in \{0, 1\}^n$ denote its incidence vector: $\chi^S_v = 1$ if $v \in S$, and $\chi^S_v = 0$ otherwise. Let $\mathcal{T}^f_G$ denote the total f-dominating set polytope, i.e. the convex hull of all the incidence vectors of the total f-dominating sets in G. Then, problem $[MWT_f]$ can be reformulated as the linear program: $\min\{c^t x : x \in \mathcal{T}^f_G\}$.

Optimization problems involving dominating sets and some of their many variants arise in several important applications, in particular for the strategic placement of resources in network infrastructures (see e.g. [13,14]). Consider a graph whose node set corresponds to locations where some resource (energy, data, …) can be made available at some cost, and whose

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edges represent connections allowing the distribution of this resource between pairs of locations. Then, an optimal solution to [MWT_f] may be interpreted as a set of locations where the resource is made available so that each location v can get it from at least f_v neighboring places and the total cost for locating the resource is minimized. For information on domination and many of its variants, the reader may consult both books by Haynes et al. [13,14], and for total domination we may refer to the survey by Henning [15] and to the book by Henning and Yeo [17]. Many works on total domination focus on finding the minimum cardinality of a total dominating set in a given graph $G = (V, E)$, i.e. the case $f_v = c_v = 1$, for all $v \in V$.

Let [DT] (resp. [DD]) stand for the decision problem associated with the minimum cardinality total dominating set problem (resp. the minimum cardinality dominating set problem). [DD] was shown to be NP-complete [12] for undirected path graphs in [5] using a reduction from the 3-dimensional matching problem. A variation of this reduction was used in [22] to prove the same result holds for [DT]. Further graph classes for which [DT] is known to be NP-complete include, e.g., split (and thus also chordal) graphs [21], line graphs of bipartite graphs [23] and circle graphs [18]. Connections between [DD] and [DT] are investigated in [20] which presents a linear time many-one reduction from [DT] to [DD]. This transformation allows the derivation of complexity results for one of the two decision problems from complexity results on the other for some particular graph families (closed for the graph transformation that is introduced there), among which the fact that the minimum cardinality total dominating set problem can be solved in polynomial time in permutation graphs, dually chordal graphs and k-polygon graphs. Laskar et al. [22] gave the first linear time algorithm to find a minimum cardinality total dominating set in a tree. Their greedy algorithm uses a particular node labeling and iteratively processes a leaf and removes it from the current tree, which is initialized with the input graph. In this paper, we extend their result by showing [MWT_f] can be solved in linear time for trees (see Proposition 11). Other graph classes for which the minimum cardinality total dominating set problem can be solved in polynomial time include strongly chordal graphs [7] and cocomparability graphs [19]. In [3] a $\mathcal{O}(n \log n)$ algorithm is presented for solving the minimum weight total dominating set problem in interval graphs. A notable graph family for which the complexity status of the problem [DD] differs from the one of [DT] is that of chordal bipartite graphs: when restricted to this graph family [DD] is NP-complete whereas [DT] can be solved in polynomial time [10].

Let $\gamma_t(G)$ (resp. $\gamma_{t,f}(G)$) denote the minimum cardinality of a total dominating set (resp. total f-dominating set) in a graph $G = (V, E)$. Given the complexity of the problem for computing $\gamma_t(G)$, some works focused on getting bounds. Lower and upper bounds on $\gamma_t(G)$ appear in [8,17]. To the best of the authors' knowledge, a lower bound on $\gamma_{t,f}(G)$ only appears in [27], while the upper bound $\frac{6n}{7}$ is reported in [16] for the particular case $f_v = 2$, for all $v \in V$.

The polyhedral structures of polytopes related to domination problems seem to have received little attention. With respect to the classical domination concept relevant works on such aspects are namely [6,11]. Let $\mathcal{D}_G$ denote the dominating set polytope, i.e. the convex hull of the incidence vectors of the dominating sets in G. Farber's work [11] gives a complete description of $\mathcal{D}_G$ for strongly chordal graphs, while Bouchakour and Mahjoub's paper [6] provides properties and characterizations of facet-defining inequalities, and also presents a peculiar decomposition result which may be formulated as follows. If $G = (V, E)$ is the 1-sum of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ (i.e. $V = V_1 \cup V_2$, $E = E_1 \cup E_2$ and $|V_1 \cap V_2| = 1$), then a complete formulation of $\mathcal{D}_G$ can be deduced from the ones of $\mathcal{D}_{G_1}$ and $\mathcal{D}_{G_2}$. We proceed to similar investigations w.r.t. $\mathcal{T}_f^G$, which, to our knowledge, do not appear elsewhere in the literature.

The paper is organized as follows. In Section 2, we present basic polyhedral results on $\mathcal{T}_G^f$. In Section 3, we prove that if the graph G has an articulation point u whose degree equals the number of connected components of the graph induced by $V \setminus \{u\}$, then an extended formulation of $\mathcal{T}_G^f$ can be obtained from complete formulations related to these components. In Section 4, complete formulations of $\mathcal{T}_G^f$ for some special graph families are given, namely: totally unimodular graphs and cycles. Then, in Section 5, a linear time dynamic programming algorithm to solve [MWT_f] for trees is presented, before we conclude in Section 6.

2. Basic polyhedral results on $\mathcal{T}_G^f$

Let $G = (V, E)$ denote a simple undirected graph, and let $f \in \mathcal{F}_G$ such that $f_v < d_v$, for all $v \in V$. In this section we give basic polyhedral properties like dimension and facet-defining inequalities of $\mathcal{T}_G^f$.

Proposition 1. *The following statements hold.*

(i) The polytope $\mathcal{T}_G^f$ has dimension n, i.e. it is full dimensional.
(ii) The trivial inequality $x_v \geq 0$ is facet-defining for $\mathcal{T}_G^f$ if and only if $f_w < d_w - 1$, for all $w \in N(v)$ such that $d_w \geq 2$.
(iii) The inequality $x_v \leq 1$ is facet-defining for $\mathcal{T}_G^f$, for all $v \in V$.

Proof. Result (i) follows from the affine independence of the incidence vectors of the following total f-dominating sets: V and $V \setminus \{v\}$, for all $v \in V$. Statement (iii) can be deduced from the affine independence of the incidence vectors of the sets: V and $V \setminus \{w\}$, for all $w \in V \setminus \{v\}$. We now prove (ii). Let $F_u^\alpha = \mathcal{T}_G^f \cap \{x \in \mathbb{R}^n : x_u = \alpha\}$ for $u \in V, \alpha \in \{0, 1\}$.

[$\Rightarrow$] In case $f_w = d_w - 1$ for some $w \in N(v)$ with $d_w \geq 2$, then necessarily $F_v^0 \subset \cap_{u \in N(w) \setminus \{v\}} F_u^1$, thus the inequality $x_v \geq 0$ cannot define a facet of $\mathcal{T}_G^f$.

[$\Leftarrow$] The incidence vectors of the n total f-dominating sets: $V \setminus \{v\}$ and $V \setminus \{v, w\}$, for all $w \in V \setminus \{v\}$ are affinely independent and they all belong to F_v^0. □