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Discrete differentiators based on sliding modes☆

Jean-Pierre Barbot[a], Arie Levant[b,*], Miki Livne[c], Davin Lunz[d]

[a] Quartz EA 7393, ENSEA, Cergy-Pontoise, France
[b] School of Mathematical Sciences, Tel-Aviv University, Israel
[c] Israel Aerospace Industries, Israel
[d] Mathematical Institute, University of Oxford, UK

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ABSTRACT

Sliding-mode-based differentiation of the input $f(t)$ yields exact estimations of the derivatives $\dot{f},\ldots,f^{(n)}$, provided an upper bound $L(t)$ of $|f^{(n+1)}(t)|$ is available in real-time. In practice it involves discrete sampling and numerical integration of the internal variables between the measurements. Accuracy asymptotics of different discretization schemes are calculated for discrete noisy sampling, whereas sampling and integration steps are independently variable or constant. Proposed discrete differentiators restore the optimal accuracy asymptotics of their continuous-time counterparts. Event-triggered sampling is considered. Extensive numeric experiments are presented and analyzed.

1. Introduction

Differentiation of noisy signals is usually performed by the algebraic, functional-analysis (Mboup, Join, & Fliess, 2009) and control/observation methods. The observation-approach is to approximate the input by a signal with known derivatives to be considered as derivatives' estimations. Such tracking is often based on high-gain (Atassi & Khalil, 2000), homogeneous (Perruquetti, Floquet, & Moulay, 2008) and sliding-mode (SM) control (Edwards, Spurgeon, & Patton, 2000; Spurgeon, 2008; Utkin, 1992; Yu & Xu, 1996). High-order sliding modes (HOSMs) (Barbot, Boutat, & Floquet, 2009; Bartolini, Pisano, Punta, & Usai, 2003; Dinuzzo & Ferrara, 2009; Feng, Yu, & Man, 2002; Fridman, Shtessel, Edwards, & Yan, 2008; Levant, 2003; Li, Du, & Yu, 2014; Moreno & Osorio, 2012; Pisano & Usai, 2011; Plestan, Glumineau, & Laghrouche, 2008; Shtessel, Taleb, & Plestan, 2012; Torres-González, Sanchez, Fridman, & Moreno, 2017) require finite-time (FT) exact robust differentiators and use homogeneity theory for their development (Angulo, Moreno, & Fridman, 2013; Bernuau, Efimov, Perruquetti, & Polyakov, 2014; Levant, 2003, 2005; Moreno, 2014; Polyakov, Efimov, & Perruquetti, 2015; Shtessel & Shkolnikov, 2003).

Homogeneous SM-based differentiators (Levant, 2003; Levant & Livne, 2012) provide for the FT exact estimation of the derivatives $f^{(i)}$, $i \le n$, of the input $f(t)$, provided an upper bound L, $|f^{(n+1)}| \le L$, is available. They also provide for the optimal error asymptotics with respect to the noise magnitude (Levant, Livne, & Yu, 2017) (see Section 3.1). Differentiators (Levant & Livne, 2019; Levant & Yu, 2018) also reject *unbounded* noises of small average value. Variable $L(t)$ is considered in Castillo, Fridman, and Moreno (2018) and Levant and Livne (2012, 2018).

A practical SM-based differentiator is a computer-based system with a noisy discretely-sampled continuous-time input, and numerical integration of the discontinuous dynamics over each sampling interval (Livne & Levant, 2014; Reichhartinger, Spurgeon, Forstinger, & Wipfler, 2017). Its error dynamics are in fact hybrid (Livne & Levant, 2014; Tuna & Teel, 2006).

The widely used Matlab solvers are based on the Runge–Kutta methods and are not applicable to SM-based dynamics due to accuracy deterioration and slow calculation. Thus the Euler method becomes the main integration method in application and simulation of such systems.

One naturally expects the vanishing Euler integration step to restore the optimal error asymptotics (Levant, 2003, 2005) obtained in the continuous-time case. That expectation is mathematically true, but we prove here that *it is practically impossible to choose a sufficiently small integration step if the differentiation order n exceeds* 1.

Novelty. This paper is the first regular publication analyzing the influence of intermediate integration steps in the discrete SM-based differentiation. We prove and demonstrate some of the results briefly announced at the conference (Barbot, Levant,

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* Corresponding author.
E-mail addresses: barbot@ensea.fr (J.-P. Barbot), levant@tauex.tau.ac.il (A. Levant), MLivne87@iai.co.il (M. Livne), davin.lunz@maths.ox.ac.uk, davin.lunz@mansfield.ox.ac.uk (D. Lunz).

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Livne, & Lunz, 2016) and, without proofs, in the survey book chapter (Levant, Levant, & Lunz, 2016). In contrast to Barbot et al. (2016) and Levant, Levant, and Lunz (2016) we also introduce new homogeneous-discretization methods, consider variable parameter $L(t)$ for differentiators and extend the results to the non-homogeneous hybrid differentiators recently introduced in (Levant & Livne, 2018).

The proposed new methods of homogeneous discretization significantly extend results of Livne and Levant (2014) and restore the optimal continuous-time accuracy asymptotics (Levant, 2003; Levant & Livne, 2012; Levant et al., 2017) for considered differentiator types. In that particular case the intermediate integration steps are shown to neither destroy nor improve the accuracy asymptotics.

A special implementation case corresponds to the input produced by an event-triggered sensor, since the sampling-time intervals become unbounded. Differentiation of such signals by SM-based technique is a long-standing problem. As a solution we propose a simple virtual-measurements' strategy removing the possible differentiation instability and even providing for the optimal accuracy asymptotics.

The paper structure. The weighted homogeneity theory and SM-based differentiators are briefly introduced in Sections 2, 3. Theoretical results are presented in Sections 4, 5. Extensive numeric experiments are analyzed in Section 6. All proofs are concentrated in appendices.

Notation. Denote $\lfloor A\rceil^B = |A|^B \operatorname{sign} A$ if $B > 0$ or $A \neq 0$; $\lfloor A\rceil^0 = \operatorname{sign} A$. Let $f(\Omega) = \{f(\omega) \mid \omega \in \Omega\}$ for any set Ω and function f. For any sets Ω, Θ and the binary operation $\diamond$ define $\Omega \diamond \Theta = \{\omega \diamond \theta \mid \omega \in \Omega, \theta \in \Theta\}$, also $\omega \diamond \Theta = \{\omega\} \diamond \Theta$.

$\|\cdot\|$ is the Euclidean norm, $B_\varepsilon = \{x| \ \|x\| \leq \varepsilon\}$. The upper semi-continuity of a compact-set function $F(x)$, $F: \mathbb{R}^k \to 2^{\mathbb{R}^k}$, means that the maximal distance from the points of $F(x)$ to the set $F(y)$ tends to zero, as $x \to y$.

A statement is said to hold *for sufficiently small (large)* $v_1, \ldots, v_k > 0$, if there exist such $w_1, \ldots, w_k > 0$ that it holds for any $v_1 \leq w_1, \ldots, v_k \leq w_k$ (respectively $v_1 \geq w_1, \ldots, v_k \geq w_k$).

2. Weighted homogeneity of differential inclusions

Let $T_x\mathbb{R}^{n_x}$ denote the tangent space to $\mathbb{R}^{n_x}$ at the point x. Recall that a solution of a differential inclusion (DI)

$$\dot{x} \in F(x),\ \ x \in \mathbb{R}^{n_x},\ \ F(x) \subset T_x\mathbb{R}^{n_x}, \tag{1}$$

is defined as any locally absolutely continuous function $x(t)$, satisfying the DI for almost all t. DI (1) is called *Filippov DI*, if $F(x)$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function.

Filippov DIs feature existence and extendability of solutions, but not the solution uniqueness (Filippov, 1988).

Introduce the weights $m_1, m_2, \ldots, m_{n_x} > 0$ of the coordinates $x_1, x_2, \ldots, x_{n_x}$ in $\mathbb{R}^{n_x}$. Define the dilation $d_\kappa(x) = (\kappa^{m_1}x_1, \kappa^{m_2}x_2, \ldots, \kappa^{m_{n_x}}x_{n_x})$ for $\kappa \geq 0$.

Recall (Bacciotti & Rosier, 2005) that a function $f: \mathbb{R}^{n_x} \to \mathbb{R}^m$ is said to have the homogeneity degree (weight) $q \in \mathbb{R}$, $\deg f = q$, if the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds for any x and $\kappa > 0$. We do not distinguish between the weight of the coordinate x_i and the homogeneity degree of the coordinate function $c_{x_i}(x) = x_i$: $\deg c_{x_i} = \deg x_i = m_i$.

A vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ (DI (1)) is called *homogeneous of the degree* $q \in \mathbb{R}$, $\deg F = q$, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$ (Levant, 2005).

Hence, the homogeneity of the vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ implies the invariance of DI (1) with respect to the combined time-coordinate transformation $(t, x) \mapsto (\kappa^{-q}t, d_\kappa x)$, $\kappa > 0$, where $-q$ can be considered as the weight of t, $\deg t = -q$.

The standard definition (Bacciotti & Rosier, 2005) of homogeneous differential equations is a particular case here. Note the difference between the homogeneity degree of a vector function taking values in $\mathbb{R}^{n_x}$ and of a vector field which takes the values in the tangent space $T\mathbb{R}^{n_x}$.

The non-zero homogeneity degree q of a vector-set field can always be scaled to ± 1 by an appropriate proportional change of the coordinate weights $m_1, \ldots, m_{n_x}$.

The *contractivity* (Levant, 2005) of the homogeneous Filippov DI (1) is equivalent to the existence of $T > 0$, $R > r > 0$, such that for all solutions $\|x(0)\| \leq R$ implies $\|x(T)\| \leq r$.

A Filippov DI $\dot{x} \in \tilde{F}(x)$ is called a *small homogeneous perturbation* of (1) if $\deg F = \deg \tilde{F}$, and $F(x) \subset \tilde{F}(x) + B_\varepsilon$, $\tilde{F}(x) \subset F(x) + B_\varepsilon$ hold for some small $\varepsilon \geq 0$ and any $x \in B_1$.

Theorem 1 (*Levant, Efimov, et al., 2016; Levant & Livne, 2016*)**.** *Let the Filippov DI (1) be homogeneous,* $\deg F = q$*. Then its asymptotic stability and contractivity features are equivalent and robust to small homogeneous perturbations. If* $q < 0$ *the asymptotic stability implies the FT stability. Moreover, the FT stability of (1) implies that* $q < 0$*.*

3. SM-based differentiation

Assumption 1. a: The input $f(t) = f_0(t) + \eta(t)$ consists of a bounded Lebesgue-measurable noise $\eta(t)$ and an unknown basic signal $f_0(t)$ with the locally Lipschitzian nth derivative satisfying $|f_0^{(n+1)}| \leq L_0(t)$ for almost all t and a locally absolutely continuous function $L_0(t) > 0$. **b:** The ratio η/L_0 is bounded, $|\eta|/L_0(t) \leq \varepsilon$. The number $\varepsilon \geq 0$ is unknown.

Assumption 2. In its turn $L_0(t)$ is provided by the additional input $L(t)$, $L(t) > 0$, $L(t) = L_0(t) + \eta_L(t)$, where $\eta_L(t)$ is a Lebesgue measurable noise, $|\eta_L(t)|/L_0(t) \leq \varepsilon_L$, and $L_0(t) > 0$, $|\dot{L}_0(t)|/L_0(t) \leq M$. The number $M \geq 0$ is known, $\varepsilon_L \in [0, 1)$ is unknown.

The problem is to evaluate the derivatives $f_0^{(i)}(t)$, $i = 0, 1, \ldots, n$, in real time.

For example, in the case of the gain-scheduled control (e.g. in flight control), when the system with the output $f(t)$ is locally approximated by linear models, $L(t)$, M are roughly determined by the model matrices and the control. The corresponding function $L(t)$ is discontinuous.

In the case of constant L we assume that $\varepsilon_L = 0$, $L = L_0$.

3.1. Homogeneous SM-based differentiators

In this subsection we assume that $L = L_0$ *is constant,* $M = 0$*.* The following is the recursive form of the differentiator (Levant, 2003). Its outputs z_j estimate the derivatives $f_0^{(j)}$, $j = 0, \ldots, n$, in FT for constant $L = L_0$, $M = 0$, $\varepsilon_L = 0$:

$$\begin{aligned}
\dot{z}_0 &= -\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t)\rceil^{\frac{n}{n+1}} + z_1, \\
\dot{z}_1 &= -\lambda_{n-1} L^{\frac{1}{n}} \lfloor z_1 - \dot{z}_0\rceil^{\frac{n-1}{n}} + z_2, \\
&\ldots \\
\dot{z}_{n-1} &= -\lambda_1 L^{\frac{1}{2}} \lfloor z_{n-1} - \dot{z}_{n-2}\rceil^{\frac{1}{2}} + z_n, \\
\dot{z}_n &= -\lambda_0 L \operatorname{sign}(z_n - \dot{z}_{n-1}).
\end{aligned} \tag{2}$$

An infinite sequence of parameters $\lambda_i > 0$ can be built starting from any $\lambda_0 > 1$, which is valid for *all* natural n (Levant, 2003). In particular, one can choose $(\lambda_0, \ldots, \lambda_7) = (1.1, 1.5, 2, 3, 5, 7, 10, 12)$ (Levant et al., 2017) which is enough for $n \leq 7$. In the absence of noises the differentiator provides for the FT exact estimations.