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Research paper

Transport in Hamiltonian systems with slowly changing phase space structure

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ABSTRACT

Transport in Hamiltonian systems with weak chaotic perturbations has been much studied in the past. In this paper, we introduce a new class of problems: transport in Hamiltonian systems with slowly changing phase space structure that are not order one perturbations of a given Hamiltonian. This class of problems is very important for many applications, for instance in celestial mechanics. As an example, we study a class of one-dimensional Hamiltonians that depend explicitly on time and on stochastic external parameters. The variations of the external parameters are responsible for a distortion of the phase space structures: chaotic, weakly chaotic and regular sets change with time. We show that theoretical predictions of transport rates can be made in the limit where the variations of the stochastic parameters are very slow compared to the Hamiltonian dynamics. Exact asymptotic results can be obtained in the one-dimensional case where the Hamiltonian dynamics is integrable for fixed values of the parameters. For the more interesting chaotic Hamiltonian dynamics case, we show that two mechanisms contribute to the transport. For some range of the parameter variations, one mechanism -called "transport by migration with the mixing regions" - is dominant. We are then able to model transport in phase space by a Markov model, the local diffusion model, and to give reasonably good transport estimates.

1. Introduction

Transport in Hamiltonian systems and is a classical field of dynamical system theory [1–4], with a huge number of applications [5,6]. Beyond deterministic dynamical systems, a lot of work has been devoted in the past to study the effect of random perturbations [7], more specifically on Hamiltonian systems and on area preserving maps, for instance in the context of plasma physics [8,9]. Noise is always present in real natural systems and in experiments because of the effect of hidden chaotic degrees of freedom. Even of small amplitude, noise plays a very important role in the long term behavior of the dynamics, and on transport properties. One usually models the hidden degrees of freedom by an additional stochastic process of small amplitude acting on the system. For example, the effect of noise on the standard map or on other classical area preserving maps has been studied earlier in [2,10,11], motivated by the dynamics of charged particles in accelerators. More recently, with the development of stochastic calculus, Freidlin and Wentzell [12,13] have studied the generic effect of small stochastic perturbations of Hamiltonian flows, and [14,15] have derived a diffusion equation for the slow action

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variable in Hamiltonian systems. In particular, Freidlin and Wentzell [12,13] have rigorously justified the averaging principle in Hamiltonian systems and studied the slow diffusive motion of action variables. The important point is that all those works fall in the dynamical framework

$$\dot{x} = \frac{1}{\epsilon}F(x) + \beta\left(x, \frac{t}{\epsilon}\right), \tag{1}$$

where F is a Hamiltonian vector field, β is a deterministic or a stochastic perturbation of the Hamiltonian vector field, and ϵ is a small parameter. Qualitatively, we can describe the dynamics of (1) saying that it follows the regular or chaotic orbits of the unperturbed dynamics $\dot{x} = F(x)$ on a fast timescale $\propto\epsilon$, and deviates slowly from those orbits because of the effect of the perturbation β, acting on a timescale of order one. We note that the case (1) where F is a Hamiltonian dynamics and β is a wave of slow modulated frequency has been studied in [16–19] and can lead to interesting phenomena such as autoresonant motion and acceleration of particles.

In this paper, we consider a different framework, which cannot be reduced to the much-studied model (1), and still is essential for many applications. We study a Hamiltonian dynamics which Hamiltonian depends on a slow parameter[1]

$$\dot{x} = \frac{1}{\epsilon}J\nabla H(x, \nu(t)), \tag{2}$$

where x represents the vector of canonical variables, and $J := \begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$.

This model has received relatively little attention in the literature compared to Eq. (1), but some nontrivial features have already been emphasized in [20,21] such as the existence of trajectories with unbounded energy growth. Generally the dynamics (1) can be very different from the dynamics (2) because the variations of ν in Hamiltonian (2) can range over a region of order one. The amplitude of the variations of ν can thus be of the same order as the variations of action variables. We call ν a "slow variable", because one has to wait for a time Δt of order one to observe a variation $\Delta\nu$ of order one, whereas canonical variables have large variations on a timescale $\propto\epsilon$. For any fixed value of the parameter ν, for the Hamiltonian $H(x, \nu)$, the dynamics in its phase space is characterized by strongly chaotic regions, weakly chaotic regions, and in some cases, KAM tori. We call phase space structure the geometry and topology of these chaotic, weakly chaotic, and regular areas. When the parameter ν slowly evolves with time, the geometry and topology of those regions are slowly distorted, and can be dramatically changed for large variations of ν that occur on long times. As a consequence of the distortion of the phase space structure, some regions that might not have been accessible for the system at the initial value of ν become easily accessible when ν changes. This affects drastically the transport in phase space. Let us consider a simple example to illustrate such a change in phase space structure. We take the one-and-a-half degree of freedom Hamiltonian[2]

$$H(p, q, \Lambda, \lambda, \nu) = \frac{p^2}{2} + \cos(q) + \cos(q - \lambda) + \nu\Lambda, \tag{3}$$

where ν is the frequency of the angle λ and plays the role of the external parameter and is the conjugated angle to Λ. In the Hamiltonian (3), a resonance is defined as the value of p for which one of the two angles q or $q + \lambda$ has zero frequency. We have plotted in Fig. 1a snapshot of the phase space structure (p, q) for the Hamiltonian (3) for three different values of ν. The frequency ν is decreasing from the left picture to the right picture. The Poincaré section described by (3) displays two major libration regions, one centered at $p = 0$, and one centered at $p = \nu$. A libration region is defined as a region of phase space where the canonical angle q has bounded oscillations between two extremal values in $[0, 2\pi]$. The libration regions can be very easily recognized on the Poincaré section of Fig. 1 as they look like " cat's eyes" surrounding the fixed points of resonances. The Hamiltonian flow is only very weakly chaotic in the initial state because the two main resonances are far away from each other, but it becomes more and more chaotic as the two main resonances become closer (for a precise description of such systems, see [2]). The red curves in Fig. 1 are particular trajectories that start inside the upper libration region. For large enough values of ν, the trajectories are trapped inside the upper libration region (first and second panel of Fig. 1). On the contrary, when ν is lower than some critical value, the trajectory can freely transit from one cat's eye to the other (third panel of Fig. 1). Imagine now that the frequency ν in Hamiltonian (3) is no longer fixed, but slowly depends on time. Consider then the Hamiltonian

$$H(p, q, \Lambda, \lambda, \nu(t)) = \frac{p^2}{2} + \cos(q) + \cos(q - \lambda) + \nu(t)\Lambda. \tag{4}$$

The three pictures of Fig. 1 represent instantaneous snapshots of the phase space of the Hamiltonian (4) at three different times. The upper cat's eye centered at $p = \nu$ moves according to the variations of $\nu(t)$. Some trajectories that were trapped in the upper libration region in the initial state can be carried downward by the displacement of the upper libration region and finally reach the lower libration region.

[1] Note that we could have defined a new Hamiltonian $H' = H/\epsilon$ such that Eq. (2) are indeed Hamilton's equations with H', but we believe the presentation chosen in this paper emphasizes more clearly the timescale separation in the dynamics.

[2] Note that we call this model one and a half degree of freedom Hamiltonian because Λ do not play any role in the dynamics and could be replaced by an explicit time dependance.