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Local continuity and asymptotic behaviour of degenerate parabolic systems

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ABSTRACT

We study the local Hölder continuity and the asymptotic behaviour of solution, $\mathbf{u} = (u^1, \ldots, u^k)$, of the degenerate system

$$u^i_t = \nabla \cdot \left(m\, U^{m-1} \nabla u^i \right) \qquad \text{for } m > 1 \text{ and } i = 1, \ldots, k$$

which describes the population densities of k-species whose diffusions are determined by their total population density $U = u^1 + \cdots + u^k$. For the local Hölder continuity, we adopt the intrinsic scaling and iteration arguments of DeGiorgi, Moser, and DiBenedetto. Under some regularity conditions, we also prove that the population density of ith species with the population M_i converges in C^∞_s to the function $\frac{M_i}{M}\mathcal{B}_M(x,t)$ as $t \to \infty$ where $\mathcal{B}_M$ is the Barenblatt profile of the standard porous medium equation with L^1 mass $M = M_1 + \cdots + M_k$. As a consequence of asymptotic behaviour, it is shown that each density becomes a concave function after a finite time.

1. Introduction and main results

For a given number $k \in \mathbb{N}$, let $u^i \geq 0$, $(i = 1, \ldots, k)$, represent the population density of ith species in a closed system and U be the total density of all species, i.e.,

$$U = u^1 + u^2 + \cdots + u^k = \sum_{i=1}^{k} u^i. \tag{1.1}$$

As a simplest case, we consider a system whose diffusion coefficients are controlled by the total population density U, i.e., let $\mathbf{u} = \left(u^1, \ldots, u^k\right)$ be a solution of

$$u^i_t = \nabla \cdot \left(m\, U^{m-1} \nabla u^i \right) \qquad \text{for } m > 1 \text{ and } i = 1, \ldots, k. \tag{SPME}$$

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By the relation (1.1), U satisfies the standard porous medium equation (shortly, PME)

$$U_t = \sum_{i=1}^{k} \left(u^i\right)_t = \sum_{i=1}^{k} \nabla \cdot \left(m\, U^{m-1} \nabla u^i\right) = \nabla \cdot \left(m\, U^{m-1} \nabla U\right) = \triangle U^m \qquad \forall (x,t) \in \mathbb{R}^n \times (0,\infty)$$

and

$$u^i(x,t) \le U(x,t) \qquad \forall (x,t) \in \mathbb{R}^n \times [0,\infty)\,,\ i = 1,\dots,k. \tag{1.2}$$

Moreover, by the regularity theory for the standard PME it is well known that the function U is locally bounded by $\|U(x,0)\|_{L^1(\mathbb{R}^n)}$ in $\mathbb{R}^n \times (0,\infty)$ (see Lemma 2.1 for details), i.e., there exists a constant $C > 0$ such that

$$\left|u^i(x,t)\right| \le |U(x,t)| \le C \frac{\|U(x,0)\|_{L^1(\mathbb{R}^n)}}{t^{\frac{n}{n(m-1)+2}}}, \qquad \forall (x,t) \in \mathbb{R}^n \times (0,\infty)\,. \tag{1.3}$$

Since both u^i and U satisfy the same equation, the relation (1.2) can be obtained by their initial conditions, i.e.,

$$u^i(x,0) \le U(x,0) \qquad \forall x \in \mathbb{R}^n,\ i = 1,\dots,k.$$

Let $0 \le U_0 \in L^1(\mathbb{R}^n) \cap L^{1+m}(\mathbb{R}^n)$ be a compactly supported function. As a general case which covers above situation, we are going to study the local continuity and asymptotic behaviour of the problem

$$\begin{cases} u_t = \nabla \cdot \left(m\, U^{m-1} \nabla u\right) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R}^n \end{cases} \tag{PME_u}$$

in the range of exponents $m > 1$, with initial data u_0 satisfying

$$0 \le u_0(x) \le U_0(x) \qquad \forall x \in \mathbb{R}^n \tag{1.4}$$

where the diffusion coefficient U is the solution of

$$\begin{cases} U_t = \nabla \cdot \left(m\, U^{m-1} \nabla U\right) = \triangle U^m & \text{in } \mathbb{R}^n \times (0,\infty) \\ U(x,0) = U_0(x) & \forall x \in \mathbb{R}^n. \end{cases} \tag{PME}$$

If U is equivalent to the solution u in the sense that

$$U(x,t) = c u^{\beta}(x,t) \qquad \forall (x,t) \in \mathbb{R}^n \times [0,\infty)$$

for some constants $c > 0$ and $\beta \in \mathbb{R}^+$, then the equation in (PME_u) appears in many physical phenomenons [1,9,28]. When $\beta(m-1)+1 > 1$, it is well known as the porous medium equation which arises in describing the flow of an ideal gas through a homogeneous porous medium [1]. Since $\beta(m-1) > 0$, the porous medium equation becomes degenerate when $u = 0$ and this degeneracy let the flow propagate slowly with finite speed. This implies that there exists an interface or free boundary which separates the regions where $u > 0$ from the regions where $u = 0$, [28]. When $\beta(m-1)+1 = 1$ and $\beta(m-1)+1 < 1$, we call them the heat equation and the fast diffusion equation, respectively. Similar to the porous medium equation, the fast diffusion equation arises in many famous flows such as Yamabe flow and Ricci flow. we refer the readers to the papers [11] for Yamabe flow and to the papers [29] for Ricci flow.

There are many studies on the regularity and asymptotic behaviour for the porous medium and fast diffusion equations. We refer the readers to the papers [3–5,7,12,14,19,24] for regularity and to the papers [2,8,15–17,27] for asymptotic behaviours of solutions of porous medium and fast diffusion equations.

Corresponding to the porous medium type equation, we can also derive the p-Laplacian equation from (PME_u) by considering the diffusion coefficients U^{m-1} as the gradients of the solution, i.e.,

$$U^{m-1} = c\,|\nabla u|^{p-2} \qquad \text{in } \mathbb{R}^n \times [0,\infty)$$