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Verification theorems for models of optimal consumption and investment with annuitization

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ABSTRACT

One can find it challenging to deal with verification theorems for optimal consumption and investment problems with annuitization. I pose a tractable framework to prove verification theorems for the problems. I revisit an annuitization model of Park (2015) and prove the verification theorem for the model. A key idea of the proof is the application of the variational inequality approach of Bensoussan and Lions (1982) to the annuitization problem solved by the suggested value function. Further, I obtain analytic comparative statics for the optimal consumption and investment strategies with annuitization.

1. Introduction

Optimal life-cycle consumption and investment decisions with annuitization have received full attention in political economy, economics and finance spheres. Indeed, annuitization is arguably one of the most important life-cycle decisions just as consumption and investment in order to achieve successful retirement.

Proving verification theorems for models with annuitization can be a considerable challenge because the question of existence of optimal strategies with discretionary stopping seems to be tricky to be answered. In addition, an expectation associated with annuitization is, in general, very difficult to compute. Compared to duality theory of Karatzas and Wang (2000), I pose a more tractable framework to prove verification theorems for problems with discretionary stopping, without resorting to the state price density (of the stochastic discount factor). Using the principle of dynamic programming, the proposed approach can be readily applied to proving verification theorems for problems with annuitization in a complete or an incomplete market.

I revisit an annuitization model of Park (2015) and prove the verification theorems for the model. While Yaari (1965) and Richard (1975) suggest full and immediate annuitization in the absence of bequest motives, i.e., all savings should be annuitized at all dates,[1] this paper follows Park (2015) and thus, considers the case where individuals need to annuitize all of their wealth at one point in time. This resembles the current situation in the UK, where individuals determine when to start their retirement pension but must do so at one point in time.[2] Actually, most variable annuity contracts in the US also provide individuals with an option to annuitize that can only be exercised once. My analysis would cover social security as well. In social security, benefits are provided in the form of a lifetime annuity based on a retirement age of 65. If individuals can opt to retire earlier (as of age 62) or later (up to age 70), they can withdraw a smaller or larger annuity, respectively.[3] In this framework, I endogenously determine the optimal timing for full annuitization and this is effectively an optimal retirement problem.[4]

As a matter of fact, the date of annuitization does not need to be always the same with the date of retirement. In line with the recent trend of optimal claiming of life annuity income even after retirement in the US, on July 2014, the US Treasury department has allowed individuals to purchase a deferred income annuity (DIA) inside tax-sheltered retirement plans. Similar to life annuity considered in this paper, to purchase the DIA, individuals

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1 Davidoff et al. (2005) suggest much weaker sufficient conditions for such full and immediate annuitization.

2 This is also akin to retirement annuity providing a stream of cash flows available for consumption during retirement years.

3 Retirement age for individuals who were born before 1960 is 66. For a detailed listing of retirement ages, please refer to www.socialsecurity.gov.

4 There is a variety of distinct strands in the portfolio choice and retirement literature. This paper sits squarely within extensions of the retirement models proposed by Choi and Shim (2006), Farhi and Panageas (2007), Choi et al. (2008), Dybvig and Liu (2010), Jang et al. (2013), and Bensoussan et al. (2016). Within the optimal portfolio and retirement choice framework, Park (2015) shows that a nonlinear option-type element that is realistically present in the actual annuity market can induce voluntary and full annuitization. In particular, such annuitization decision can be regarded as an American-style option. Individuals would rather enjoy the extra leisure derived from their voluntary annuitization than the value of income obtained from their work, as soon as they touch a certain wealth threshold over which it is optimal to annuitize all of their wealth.

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have to pay a lump sum at one distinct point in exchange for a defined (or a fixed) lifetime income. However, to purchase the DIA, individuals do not have to pay such a lump sum at only retirement. Rather, individuals can purchase it at a relatively young age and delay its income generating date until a relatively old age (usually after the normal retirement age).[5] Although I do not take the DIA into this paper explicitly, in Section 5, I partially address the optimal claiming of life annuity income in the US by considering an exponentially distributed retirement age. In this case, individuals can withdraw their annuity income before or after the normal retirement age. Ultimately, I can investigate the effects of variation in retirement time on the annuitization decision.

One unique feature of the suggested framework is the application of the variational inequality approach of Bensoussan and Lions (1982) to the annuitization problem solved by the proposed value function. I verify the uniqueness and existence for the value function. Further, I obtain analytic comparative statics for the optimal strategies. In terms of tractable applications, I hope that this paper will lend itself to the verification for many other interesting utility maximization problems with discretionary stopping.

The paper is organized as follows. In Section 2, I specify the basic model in which the annuitization model of Park (2015) is revisited. In Section 3, I prove the uniqueness and existence theorem for the model. In Section 4, I provide analytic comparative statics for the derived optimal strategies. In Section 5, I consider more general case for annuitization than Section 2, where individuals are allowed to invest in both bonds and stocks after annuitization. In Section 6, I conclude the paper.

2. The basic model

In order to propose a tractable approach to proving verification theorems for optimal consumption and investment problems with annuitization, I need to solve a realistically calibrated model of annuitization. To allow for a realistic model of annuitization in the simplest possible setting in every other dimension, I revisit an annuitization model of Park (2015).

Let me specify the financial market. The market is comprised of two broad classes of assets: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price B_t is given by

$$dB_t = rB_t dt,$$

where $r > 0$ is the risk-free interest rate. The stock price S_t follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > r$ is the expected rate of the stock return, $\sigma > 0$ is the stock volatility, and W_t is a standard Brownian motion defined on a suitable probability space.

An individual receives a constant stream of labor income I while she is working. I allow for the case where the individual can borrow against her labor income. That is, she can borrow up to the present value $I/(r+\nu)$ of labor income discounted at the sum of risk-free interest rate and mortality rate. Here, I assume that mortality rate is constant. I relax this assumption in Section 5. The individual is endowed with an amount of financial wealth $x > -I/(r+\nu)$.[6] As long as she is working, the wealth dynamics follow

$$dX_t = (rX_t - c_t + I)dt + \pi_t \sigma(dW_t + \theta dt), \;\; X_t > -\frac{I}{r+\nu}, \;\; 0 \le t < \tau, \tag{1}$$

where π is the dollar amount invested in the stock market and $\theta = (\mu - r)/\sigma$ denotes the Sharpe ratio.

As far as the model of Park (2015) is concerned, the individual aims to maximize the following life-time utility by optimally controlling her consumption c, investment π, and timing of voluntary annuitization τ:

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\left[\int_0^\tau e^{-(\beta+\nu)t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-(\beta+\nu)\tau}\frac{\bar{l}^{\gamma-\gamma^*}}{1-\gamma}\frac{\left(X_\tau(r+\nu)\right)^{1-\gamma}}{\beta+\nu}\right], \tag{2}$$

where E is the expectation taken at time 0, $\beta > 0$ is the individual's subjective discount rate, $\nu > 0$ is the individual's constant mortality rate, $\gamma^* > 0$ is the coefficient of relative risk aversion, and $\gamma \equiv 1 - a(1-\gamma^*) > 0$. Here, $0 < a < 1$ denotes a weight for consumption in the following Cobb–Douglas type utility preference:

$$U(l_t, c_t) = \frac{1}{a}\frac{(l_t^{1-a}c_t^a)^{1-\gamma^*}}{1-\gamma^*},$$

where l_t is leisure at time t, which can be regarded as the time taken away from working. I consider two cases of leisure: (1) $l_t = \underline{l}$ while the individual is working (2) $l_t = \bar{l}$ ($\bar{l} > \underline{l}$) when the individual annuitizes all of her wealth. I normalize $\underline{l} = 1$. For the notational convenience, I define

$$K \equiv \frac{\bar{l}^{\gamma-\gamma^*}}{\beta+\nu},$$

which represents the preference for leisure after voluntary annuitization.[7] I assume that there is no bequest motive.

In my optimization problem (2), the optimal timing of voluntary annuitization is determined endogenously as the date at which the entire portfolio is geared toward riskless bonds, rather than a cross between stocks and bonds. This, in fact, reflects the canonical annuity results in Yaari (1965) and Richard (1975) that it is optimal to consume the entire annuity income in the absence of bequest motives. In line with this, I assume that at some future time τ, individuals annuitize all of their wealth X_τ and consequently, consume at a rate of $X_\tau(r+\nu)$, which is the annual annuity income.[8]

3. Solutions

For a fixed stopping time τ, I define

$$J_\tau(x) \equiv \max_{(c,\pi)} E\left[\int_0^\tau e^{-(\beta+\nu)t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-(\beta+\nu)\tau}K\frac{\left(X_\tau(r+\nu)\right)^{1-\gamma}}{1-\gamma}\right].$$

5 The annuity income can be started to generate even after age 70, but before age 85. In this sense, such a DIA is also known as an Advanced Life Delayed Annuity.

6 The implies that the individual can consume and invest in the stock as long as her financial wealth is above $-I/(r+\nu)$. If the wealth level approaches $-I/(r+\nu)$, then the individual cannot consume and invest any more. In this case, consumption and risky investment must be zero.

7 For instance, the preference for leisure stems from a disutility of work, household production, and cost savings. In reality, after annuitization the individual can have sufficient time to enjoy leisure such as shopping for bargains, preparing meals, and taking a cruise etc.

8 There are two types of benefit plans for retirement. On the one hand, defined benefit plans guarantee a defined amount of income for remaining lifetimes after retirement, which are mainly considered in the paper. On the other hand, defined contribution plans (also better known as 401(k) retirement plans) provide a variable amount of income, depending on market/economy conditions. In light of the growing popularity of 401(k) retirement plans, annuities can likely to be indexed to stock returns. The current market seems to be coming around to the judgment in favor of defined contribution plans rather than defined benefit ones. To cover that situation as well, I investigate the case for which individuals can still have a great amount of flexibility after annuitization by allocating their financial resources to both bonds and stocks, instead of merely bonds. For the details, please refer to Section 5.