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Aleatory quantile surfaces in damage mechanics

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Abstract

In statistical damage mechanics, a deterministic failure limit surface is replaced with a scale-dependent family of quantile surfaces. An idealized homogeneous isotropic matrix material containing cracks of random size and orientation is used to elucidate expected mathematical character of aleatory uncertainty and scale effects for initiation of damage in a brittle material. Scope is limited to statistics and scale dependence for the ONSET (not subsequent progression) of shear-driven failure. Exact analytical solutions for probability of such failure (with an interesting pole-point visualization) are derived for axisymmetric extension or compression of a single-crack sample. A semi-analytical bound on the failure CDF is found for a multi-crack specimen by integrating the single-crack probability over an exponential crack size distribution for which the majority of flaws are small enough to be safe from failure at any orientation. Resulting tails of the predicted failure distribution differ from Weibull theory, especially in the third invariant.

Keywords: Statistical damage; Aleatory uncertainty; Scale effect; Pole point; Verification

1. Introduction

Damage initiation, *i.e.* the beginning of microcrack growth (or, more broadly, the beginning of stiffness degradation) can profoundly influence subsequent continued material damage.[1] In high-rate loading, release waves from a damage nucleation site generally lag behind initial stress waves, thereby allowing sufficiently distant regions to become damaged before the release waves can arrive. Grady,[2] Curran,[3] Hild,[4] et al. extended seminal work of Mott[5] by recognizing this "event horizon" phenomenon gives rise to a larger number of smaller fragments as loading rate increases. Difficulties implementing these theories into massively-parallel simulations of macroscale structural failure are rooted in the need for fracture to be treated as a bifurcation, which requires aleatory uncertainty to be incorporated explicitly in simulations in order to stimulate the bifurcation in a realistic, mesh-insensitive manner. Ultimate strength is well established to vary with specimen size, so there is no representative volume element (RVE) size above which continuum strength settles into a constant value.[6,7] Hence, any reported value of ceramic or rock strength is meaningless unless accompanied by information about the size of the sample used in testing as well as information about the repeatability (preferably distributions) of strength values.

Large-scale engineering damage simulations (such as blast and penetration of a dam, a building, or armored vehicle) require finite-element formulations that account for sub-scale heterogeneity. Whether modeled explicitly (as in concurrent multiscale and/or micromechanics-based simulations[8]) or implicitly (as in scale-dependent statistical smeared damage approaches[9]), microcracks must be spatially distributed with a realistic mean free path, and they must furthermore have realistic variation in size and orientation to provide a statistically variable macroscale strength and size effect in which larger specimens (and hence larger finite elements) have greater probability of failure because they are more likely to contain a critically large or critically oriented flaw.

In light of the event-horizon effect, predicting the statistics of failure *initiation* is an obvious prerequisite to reliably modeling subsequent cascading failure. Fig. 1 illustrates some possibilities for distributions of initial failure strength that have been previously explored in the literature, each differently generating random realizations of strength taken from probability "clouds" centered about scale-dependent median strength surfaces.

In the simulations of Fig. 1, each finite element is governed by a deterministic damage model that is initialized with statistically

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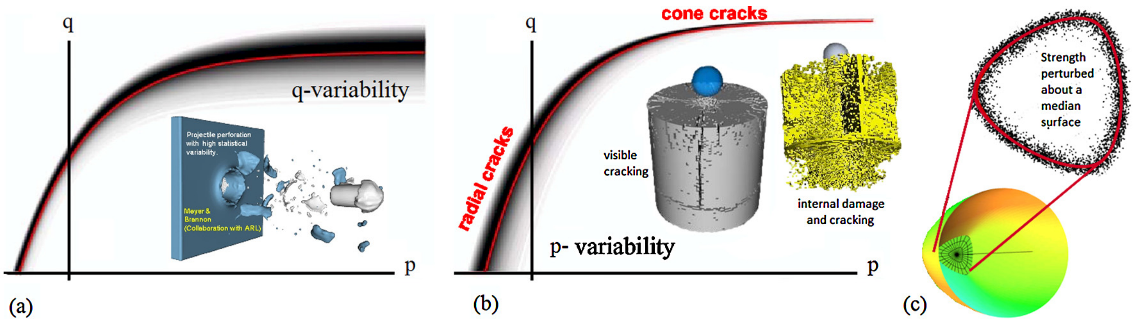

Fig. 1. Statistically varied limit surface parameters: (a) "q-variability," which perturbs high-pressure strength, has been used in metallic plate perforation,[10] (b) "p-variability," which perturbs low-pressure strength, has been applied to impact-induced ceramic damage,[9] (c) octahedral strength realizations.

variable and scale-dependent strength parameters. Statistics and scale effects were also applied to properties for subsequent rate-sensitive cascading failure, which is outside the scope of this paper's focus on failure *initiation*. Deviation from weakest link behavior is expected in the prediction of cascading failure, but not in failure initiation. Onset of material failure often doesn't lead to complete cascading failure, as should be familiar to experimentalists who can hear audible pops (acoustic emissions) during brittle strength testing long before the sample fails catastrophically. Failure *initiation* without cascading failure is also evident in the "sprinkled" (non-coalesced) failure points in the inset simulation of Fig. 1b. These act to seed perturbations leading to fractures.

Each statistical realization of the otherwise deterministic failure model initiates failure (i.e., loss of strength and stiffness) when the equivalent shear stress q in the element reaches a critical value that itself typically depends on the pressure p and the Lode angle θ_L. In terms of standard stress invariants, $I_1 = \text{tr}\boldsymbol{\sigma}$, $J_2 = \frac{1}{2}\text{tr}[(\text{dev}\,\boldsymbol{\sigma})^2]$, and $J_3 = \frac{1}{3}\text{tr}[(\text{dev}\boldsymbol{\sigma})^3]$, where dev$\boldsymbol{\sigma}$ denotes the deviatoric part of the Cauchy stress $\boldsymbol{\sigma}$, these standard invariants are

$$p = \frac{-I_1}{3}, \qquad q = \sqrt{3J_2}, \qquad \text{and}$$
$$\theta_L = \frac{-1}{3}\sin^{-1}\left[\frac{J_3}{2}\left(\frac{3}{J_2}\right)^{3/2}\right] \quad . \tag{1}$$

As labeled in Fig. 2, our definition of the Lode angle θ_L ranges from $-30°$ in triaxial extension (TXE) to $+30°$ in triaxial compression (TXC).[a] In a state of pure shear, $\theta_L = 0$. Aside from the common p and q invariants, we also use "isomorphic" (Lode) stress invariants,

$$r_L = \|\text{dev}\boldsymbol{\sigma}\| = \sqrt{\frac{2}{3}}\; q \qquad \text{and}$$
$$z_L = \frac{-I_1}{\sqrt{3}} = \sqrt{3}\;\; p \quad . \tag{2}$$

The Lode invariants (r_L, θ_L, z_L) represent cylindrical coordinates centered about the [1, 1, 1] hydrostat in 3D principal stress space, and they are sometimes called "isomorphic" invariants because lengths and angles in a 2D plot of r_L vs. z_L are the same as those in the 2D manifold of 6D stress space spanned by the identity and stress deviator tensors.

To model differences in tension and compression, strength q typically increases with p (or, equivalently, r_L increases with z_L), as illustrated in Fig. 1, where a "cap" may be added for porous media. The limit-surface envelope for a brittle or quasi-brittle material (*c.f.*, [11]) typically also has strength differences in TXE and TXC at a common pressure, which motivates having strength depend on the third stress invariant (Lode angle). In Fig. 1c, a triangular fracture-dominant octahedral profile at low pressure smoothly transitions to a more circular plasticity-dominated profile at high pressure.[12]

The simulations in Fig. 1 assign a statistically variable strength to each element, thereby producing elements that are weak (W) or strong (S) in comparison to the median. The median quantile (iso-probability) surface itself is also scale dependent so that smaller finite elements are stronger, on average.[b] The tension loading path (arrow in Fig. 2a) passes through a sequence of median strength surfaces that would be assigned to specimens of decreasing size to produce the inset plot of strength vs. size, identical in character to indirect tension data for concrete.[16] The model in Fig. 2a automatically predicts a different (more fracture-dominant) size effect for a spherical tension path (i.e., an arrow pointing directly to the left). For paths pointing towards the compression direction, this model likewise naturally predicts both higher strength and lower variability with increased confinement. This paper explores a simple microphysical basis for these trends.

Even under conditions of theoretical material instability (such as loss of positive definiteness of the tangent or acoustic tensors; *c.f.*, [17–19]), a simulation must include realistic scale-dependent perturbations to accurately predict subsequent post-instability structural response. Naïvely relying on numerical round-off to stimulate a bifurcation event will inevitably

[a] Despite its unfortunate name, triaxial compression (TXC) is actually an axisymmetric stress state (i.e., only *two* distinct eigenvalues) for which the axial stress is more compressive than the lateral stress. Triaxial extension (TXE) is also axisymmetric, but with the axial stress less compressive than the lateral stress.

[b] Scaling is essential to reduce sensitivity of failure probability to discretization. Kamojjala, *et al.* (this volume) further shows that under-resolution of nonuniform stress fields can give numerical errors as significant as physical errors attributed to neglect of long-range fluctuations in brittle systems.[13–15]