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Journal of the European Ceramic Society 34 (2014) 2737–2744

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A cracked infinite Kirchhoff plate supported by a two-parameter elastic foundation

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Available online 25 January 2014

Abstract

This paper presents a full-field solution for the linear elasto-static problem of a homogeneous infinite Kirchhoff plate with a semi-infinite rectilinear crack resting on a two-parameter elastic foundation. The same model describes the problem of a plate equi-biaxially loaded in its mid-plane by a constant normal force and, as a limiting case, the problem of a spherical shell. The full-field solution is obtained in closed form through the Wiener–Hopf method in terms of Fourier integrals. The stress-intensity factor (SIF) for the case of symmetric (K_1) and skew-symmetric (K_2) loading conditions is obtained and the role of the soil parameters is discussed. In particular, it is shown that a purely local model (Winkler) is unable to provide a safe-proof design limit.

Keywords: Cracked plate; Foundation; Wiener–Hopf; Stress-intensity factor

1. Introduction

Fracture mechanics in linear elasticity has drawn extensive attention since the pioneering works by Griffith in the mid 1950s and it has been a hot research topic ever since. There is a vast body of literature on the subject, which spans from elasto-static to elasto-dynamics, from atomistic to multi-scale approaches, from reduced dimensional to 3-D theories. The reason for such enthusiasm lies in the great scientific and industrial potential for effective *structural integrity assessment*, as a mean of determining the fitness-for-service (FFS) of a structure/material, either at manufacturing, at purchase, during or at the end of service life. This drive has prompted governments and scientific institutions towards developing reliable, cost-effective and unified protocols of integrity assessment with maximum applicability across all scales of final users, such as the EU-partly funded Structural INTegrity Assessment Procedure (SINTAP) and the Fitness-for-Service Network (FITNET).[15] A major role in such protocols, which have inspired and informed world-wide regulations such as British Standard BS 7910 and API RP 579, is played by the knowledge of the stress intensity factor K (SIF) for the geometry and loading condition at hand. Indeed, the failure of cracked components is governed by the stresses in the neighbouring of the crack tip, which is described by the SIF. Despite the availability of several handbooks for SIFs,[10,9] very few full-field solutions are available for cracked plates resting on an elastic foundation. To the authors' best knownledge, they are in fact just two. [1,2] This lack of results is problematic, since this situation often occur in practice (e.g. roadways, pavements, floorings, etc.). Furthermore, when some results are available, they never involve the foundation's mechanical properties alone. For instance, in Ref. 2 the problem of a finite crack in an infinite Kirchhoff plate supported by a Winkler foundation is considered and it is reduced to a singular integral equation. However, since two length scales exist in the problem (the crack length and the foundation relative Winkler modulus), the SIF may be related to some dimensionless ratio of them and not directly to the foundation's mechanical property. In actual facts, this outcome stems from the Winkler approximation to the foundation and not from the physical feature of the problem. In Ref. 1, the semi-infinite rectilinear crack problem for an infinite Kirchhoff plate resting on a Winkler foundation is considered and the full-field solution obtained. Since this is a self-similar problem, no characteristic length scale exists. Application of the above results is given to road and airport pavements in Ref. 5 and, more recently, in Ref. 7. As a result, the influence of the pavement foundation

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http://dx.doi.org/10.1016/j.jeurceramsoc.2013.12.029

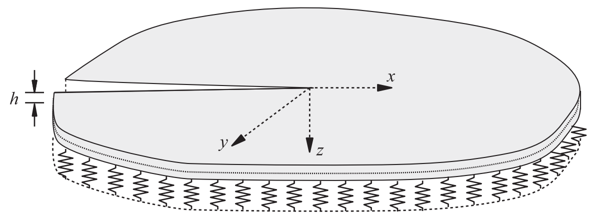

Fig. 1. Cracked Kirchhoff plate resting on a Pasternak-type elastic foundation.

on the SIF cannot be properly assessed. Several papers address crack problems in plate theory.[4,11,13,3] A literature review of the asymptotic behavior of the stress field at the crack tip is given in Ref. 14 along with comparison with the available experimental results.

This paper deals with the elasto-static semi-infinite rectilinear crack problem for an infinite Kirchhoff plate resting on a two-parameter elastic foundation under very general loading conditions. The foundation, also termed Pasternak-type, is weakly non-local, as it accommodates for coupling among the independent springs of a purely local model (i.e. Winkler model). The same model governs the problem of a Kirchhoff plate equi-biaxially loaded in its mid-plane. The Pasternak foundation accounts for two length scales such that the whole problem is governed by a parameter η expressing the soil to plate relative stiffness. Discussion is here limited to the range $\eta \in (0, 1)$. The limiting case as $\eta \to 0$ gives the Winkler model. The paper is organized as follows. Section 2 presents the problem which is then formulated in terms of a pair of dual integral equations. The latter is solved at Section 3 through the Wiener–Hopf (W–H) technique. Numerical results are given at Section 4 where SIFs are obtained and some conclusions drawn. Finally, the details of the W–H the factorization are presented in the Appendix at the end of the paper. A similar approach has been considered for Mode III crack problems in couple stress elastic materials (Refs. 6,16-18).

2. Governing equations

Let us consider a Kirchhoff infinite plate supported by a Pasternak-type two parameter elastic foundation (Fig. 1). A Cartesian reference frame is attached to the crack tip so thus the crack is located along the negative part of the x-axis. The governing equation for the transverse displacement of the plate w reads

$$D\Delta\Delta w = q - \pi, \tag{1}$$

being $\Delta = \partial_{xx} + \partial_{yy}$, the Laplace operator in two dimensions; q, the transverse distributed load; D, the plate bending stiffness and π, the soil reaction. For a Pasternak-type foundation, the latter is given by

$$\pi = kw - c\Delta w, \tag{2}$$

wherein k and c are the Winkler and the Pasternak moduli, respectively. Eqs. (1) and (2) may be rewritten as

$$\Delta\Delta w - \frac{2}{\chi^2}\Delta w + \frac{1}{\lambda^4} w = \frac{q}{D}, \tag{3}$$

having let the length scales

$$\lambda = \sqrt[4]{\frac{D}{k}}, \quad \chi = \sqrt{\frac{2D}{c}}.$$

Let the dimensionless quantities be introduced

$$(\hat{x}, \hat{y}, \hat{w}) = (x/\lambda, y/\lambda, w/\lambda),$$

together with the positive dimensionless ratio

$$\eta = \lambda/\chi = \sqrt{\frac{c}{2\sqrt{kD}}}.$$

Eq. (3) is now formally factored in terms of the dimensionless Laplacian $\hat{\Delta} = \partial_{\hat{x}\hat{x}} + \partial_{\hat{y}\hat{y}}$

$$\left(\hat{\Delta} + \alpha^2\right)\left(\hat{\Delta} + \beta^2\right)\hat{w} = \hat{q} = \lambda^3 q/D, \tag{4}$$

wherein

$$\alpha^2 = -(\eta^2 - \sqrt{\eta^4 - 1}), \quad \beta^2 = -(\eta^2 + \sqrt{\eta^4 - 1}).$$

Hereinafter, z^* denotes the complex conjugated of z. Let us assume, for the sake of definiteness, $\eta < 1$, that is $c < 2\sqrt{kD}$. Besides, let α and β to be placed in the upper complex half plane (Fig. 3), i.e.

$$\left.\begin{array}{l}\alpha\\ \beta\end{array}\right\} = \pm\sqrt{\frac{1-\eta^2}{2}} + i\sqrt{\frac{1+\eta^2}{2}}, \tag{5}$$

being $i^2 = -1$. It is observed that

$$\beta = -\alpha^* \quad and \quad |\alpha| = 1. \tag{6}$$

Hereinafter, only dimensionless quantities will be considered and the hat dropped to lighten notation. Accommodating for boundedness at $y \to \infty$, the general solution of the homogeneous ODE (4) reads

$$w = w_1 + w_2, \tag{7}$$

where

$$w_1(x, y^\pm) = \int_\gamma A_1^\pm \exp\left(-\sqrt{s^2 - \alpha^2}|y^\pm| + isx\right) \mathrm{d}s$$

and

$$w_2(x, y^\pm) = \int_\gamma A_2^\pm \exp\left(-\sqrt{s^2 - \beta^2}|y^\pm| + isx\right) \mathrm{d}s.$$

The integration path γ lies in the complex plane and it is yet to be defined (see Section 3) while the square root is taken with positive real part. It is understood that $y^+ \in [0, +\infty)$ and $y^- \in (-\infty, 0]$, whereas $A_1^\pm$, $A_2^\pm$ are four functions of s to be determined. Such functional dependence will be tacitly assumed throughout this section. Let $A_i^\pm$ split in a symmetric and skew-symmetric part

$$A_i^\pm = \overline{A}_i \pm \Delta A_i, \quad i = 1, 2, \tag{8}$$

where $\overline{A}_i$ and ΔA_i are likewise functions of s. Let the (dimensionless) bending moment and equivalent shearing force

$$m_y = -\left(\partial_{yy} + \nu\partial_{xx}\right) w, \quad v_y = -\partial_y\left[\partial_{yy} + (2-\nu)\partial_{xx}\right] w$$