Available online at www.sciencedirect.com

ScienceDirect

Journal of the European Ceramic Society 34 (2014) 2807–2821

www.elsevier.com/locate/jeurceramsoc

Feature Article

On a 3D crack tracking algorithm and its variational nature

A. Salvadori [a,b,*], F. Fantoni [a]

[a] DICATAM – Dipartimento di Ingegneria Civile, Architettura, Territorio, Ambiente e di Matematica, Università di Brescia, via Branze 43, 25123 Brescia, Italy

[b] Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Received 4 September 2013; received in revised form 26 November 2013; accepted 3 December 2013

Available online 9 January 2014

Abstract

The crack propagation problem for linear elastic fracture mechanics has been studied by several authors exploiting its analogy with standard dissipative systems theory (see e.g. Nguyen (2000), Mielke (2005) and Francfort and Marigo (1998)). In recent publications Salvadori and Carini (2011) and Salvadori and Fantoni (2013) minimum theorems were derived in terms of crack tip "quasi static velocity" for two- and three-dimensional fracture mechanics. They were reminiscent of Ceradini's theorem Ceradini (1965, 1966) in plasticity.

Such an incremental picture naturally leads to explicit methods for integration in time, with well know drawbacks in terms of accuracy and stability. The present work introduces an implicit Newton–Raphson based crack tracking algorithm which is endowed with a variational formulation.

Keywords: Fracture mechanics; Crack growth; Ceramic materials; Variational formulations

1. Introduction

Brittle materials include most ceramics, that will tend to fracture before any plastic deformation takes place, often resulting in catastrophic failures. Understanding and predicting the fracture pattern evolution may improve the mechanical performance and increase the safety of materials and components. An abundant literature was devoted in the last decades to model and control the initiation and growth of fractures in ceramic materials. Scientific outcomes impact spans a wide range of applications.

In the expansive field of renewable energy for instance there is a large interest in ceramic materials for energy harvesting and storage. The potentialities of some tailored ceramics materials [8,9] have been known for some time, but their exploitation has been until recently prevented by a serious issue associated with the large volume expansion-contraction changes experienced during the alloying-de-alloying electrochemical process, that in turn induced cracks and eventually pulverization of electrodes, finally leading them to die in the round of few cycles.

There is high interest in a better understanding and modeling of the fracture mechanisms in ballistic protection systems.[10] Ceramic layered structures primarily acts to break up and decelerate the projectile. A core of highly fragmented material is formed in the front plate beneath the point of impact, from which radial and secondary cracks emanate. Limiting these radial cracks is of interest to limit the damaged area - partly to maintain protection against further projectile impacts. The ceramic thus needs to be hard enough to erode the projectile and to decrease its velocity, but radial crack propagation after impact should ideally not be extensive.

Ferroelectric ceramics are becoming widely employed materials for actuators and sensors. As such, prediction of device performance and reliability are important technological problems. A rather lengthy literature has developed for the fracture mechanics of piezoelectric solids. According to Landis et al. [11] the current state of theory for fracture of materials in the presence of electric fields is unsatisfactory. Aside from the lack of a constitutive law for ferroelectric switching which would be required for determining the fields around a crack tip, even the linear piezoelectric theory is incomplete.

The prediction of fracture patterns is based on the knowledge of crack tracking algorithms, which is a fecund and modern topic in fracture mechanics. The present note aims at providing a contribution in this vibrant context.

* Corresponding author at: DICATAM – Dipartimento di Ingegneria Civile, Architettura, Territorio, Ambiente e di Matematica, Università di Brescia, via Branze 43, 25123 Brescia, Italy. Tel.: +39 030 3711239; fax +39 030 3711312.

E-mail address: alberto.salvadori@ing.unibs.it (A. Salvadori).

0955-2219/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.

http://dx.doi.org/10.1016/j.jeurceramsoc.2013.12.010

Fracturing processes reveal three distinct phases: loading without crack growth, stable and unstable crack propagation. Energy dissipation due to fracture growth takes place in the process region, in the plastic region outside the process region, and eventually in the wake of the process region. When a fracture process is idealized to infinitesimally small scale yielding, energy dissipation is concentrated at the crack front: this note stems from such idealization.

Similarly to the determination of the elastic limit, the concept of incipient crack growth is difficult to identify: in both cases, the difficulty is solved by a convention. An *onset of crack propagation* and a *safe equilibrium domain* are governed theoretically by a local condition on **K**, describing when the process region reaches a critical state which, in most cases of engineering interest, is independent of body, geometry and loading: this property is termed autonomy (see Ref. [12])

The global *incremental* quasi-static fracture propagation problem consists in seeking an expression of the *crack growth rate* $\dot{l}(s, t)$ at a generic point s along the crack front in the presence of a variation of the external actions for all three phases of a fracturing process. The problem can be framed in the mechanics of *standard dissipative systems* [1,2] and phrased in the following way: given the state of stress and the history of crack propagation (if any) at time t, express the crack propagation rate (if any) of the crack front due to a variation of the external actions as a function of the stress and of the history.

The *quasi-static fracture propagation* was revisited and posed as an energy (global) minimum problem. [3,13,14] Various types of crack propagation within the variational framework have been discussed in Freddi and Royer-Carfagni,[15] whereas an extension of the variational approach to the case of plastic flow has been recently attempted by Ambrosio and Lemenant.[16]

For two- and three-dimensional linear elastic fracture mechanics (LEFM), the incremental quasi static fracture propagation problem was studied by means of an analogy between crack propagation and rigid-plasticity.[17] A maximum dissipation principle at the crack tip during propagation was postulated. Associated flow rule and loading/unloading conditions in Karush–Kuhn–Tucker complementarity form descended. Minimum theorems were further derived [4,5] in terms of crack tip "quasi static velocity", in force of which crack growth rates are minimizers of linearly constrained quadratic functionals. They were reminiscent of Ceradini's theorem [6,7] in plasticity.

Ceradini's variational view on the incremental equilibrium of elastic-perfectly-plastic materials was extended further by Capurso and Maier to the linear hardening with associative and non associative flow rule,[18–20] and Mariano [21] to non-linear hardening and other non-standard classes of plastic phenomena such as strain-gradient plasticity, Cosserat plasticity, micromorphic plasticity). Using similar arguments, Authors aim at extending in the near future the analogy formulated in Salvadori,[17] so to model a broader class of fracture growth phenomena within a unique variational framework. Among the phenomena under investigations hydraulic fracturing is worth to be mentioned together with crack propagation due to thermal effect, or driven by diffusion of species in solids, cracks in elastoplastic materials,[22,23] in elastic materials with defects [24] and in couple stress elastic materials.[25]

Ceradini's functionals [6,7] have been recently revisited in terms of weight functions.[5] Moving from the map of velocities of crack elongation along the crack front, algorithms for crack advancing of the explicit type were formulated, which were driven by the increment of external actions and allowed the step-wise approximation of crack length increment at the crack tip. In their simplest formulation, those algorithms do not enforce the Griffith condition step by step and may lead to large errors in the estimation of both the critical load and configuration, measured at the transition between stable and unstable propagation regimes. Such a condition is here recalled in Section 4.

The present work introduces an implicit, Newton–Raphson based, crack tracking algorithm which is endowed with a variational formulation within each iteration. Moving again from the analogy between rigid-plasticity and LEFM, a return mapping algorithm is formulated owing to the convexity of the Maximum Energy Release Rate safe equilibrium domain. An "elastic trial" is set as usual by the increment of the external actions: owing to the linearity of the elastic problem, the evaluation of the trial state in terms of Stress Intensity Factors (SIFs) is trivial. The Griffith condition is tested for the computed SIFs and, if violated, a Newton–Raphson scheme is triggered off. To this aim, a Gateaux derivative operator $N'_{\mathcal{F}_0}$ is evaluated, which shows great similarities with the operator for the incremental problem. In particular, it is symmetric with respect to a standard bilinear form. Accordingly, at each iteration of the Newton–Raphson scheme a constrained minimum problem can be solved, which provides a new estimation of the crack front location. At such a new crack front, updates SIFs can be evaluated from the global problem. The Griffith condition is tested for the updated SIFs and, if violated, the iteration process is continued up to convergence. It is a remarkable difference between plasticity and fracture that at each iteration the geometry of the crack front, and thus of the global problem, changes.

The crack tracking framework is tested on a simple problem, the axial symmetric example of a penny-shaped crack subjected to a point-load tensile action, in Section 7. Results confirm the potential of the proposed finite-step formulation.

2. Notation and preliminaries

Consider a three dimensional body $\Omega \in \mathbb{R}^3$ and a crack of arbitrary shape as in Fig. 1, except that both its surface $\mathcal{S}$ and front $\mathcal{F}$ are assumed to be of class C^∞, at least in the vicinity of $\mathcal{F}$.[26,27] Isotropic linear elasticity on Ω is assumed in the present note, making use of Hooke's law without limitation of stress and strain magnitudes.

The material response to the following quasi-static external actions is sought: tractions $\overline{\mathbf{p}}(\mathbf{x})$ on $\Gamma_p \subset \partial\Omega$, displacements $\overline{\mathbf{u}}(\mathbf{x})$ on $\Gamma_u \subset \partial\Omega$, bulk forces $\mathbf{f}(\mathbf{x})$ in Ω. External actions are all assumed to be *proportional*, i.e. that they vary only through multiplication by a time-dependent scalar $\kappa(t)$, termed load factor,