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# Bend strength of alumina ceramics: A comparison of Weibull statistics with other statistics based on very large experimental data set

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**Abstract**

We have performed a statistical evaluation of 5100 experimental values of the bend strength of test pieces from a serial production of alumina products. The Weibull distribution was compared to three other, commonly known, 2-parametric distributions in order to reveal which of them best matches the experiments. The maximum-likelihood method was used to evaluate the corresponding parameters, and then a *Q*–*Q* plot was used for all the statistics. We confirmed that the Weibull distribution describes the experimental strengths most accurately.



*Keywords:* Bend strength; Weibull statistics; Maximum-likelihood method; *Q*–*Q* plot

## 1. Introduction

The scatter in the values of strength measured in typical mechanical tests for brittle materials, such as ceramics, is usually described by the Weibull statistical distribution, either two- or three-parametric, or the one corresponding to more fracture modes.[1–6] The reliability of the Weibull distribution has been theoretically and experimentally investigated for a very broad range of conditions.[7–23] One of the typical experimental problems is that the cost limits the number of testing pieces for the strength measurements, which makes the prediction of the free parameters in the chosen distribution less reliable.

Different calculation procedures are used to evaluate the Weibull parameters (or the corresponding parameters in other statistical distributions), the most popular being the linear-regression (LR) method and the maximum-likelihood (ML) method. Each of these methods has its benefits and drawbacks.[7,8] Monte–Carlo simulations are a very useful tool for predicting the reliability of various estimation methods and their optimization, particularly when they are combined with experiments. These simulations indicate that all of the estimation methods, such as LR and ML, show some biasing in the estimated parameters, depending on the size of the test group and the adaptation and optimization of the particular method. The maximum-likelihood method is a standard method due to its efficiency and its ease of application when censored failure populations are encountered.[24] Since this method has proved to be particularly suitable, several variants of it have been proposed and tested, for instance, the generalized maximum-likelihood method (GMLE), which uses various rank estimators.[25–27] In addition, some authors tested the idea of dividing several measured strength values of ceramic materials into random, smaller subsets in order to study the corresponding statistical distribution of the Weibull parameters.[4,5,8–10]

However, the justification for the use of the Weibull distribution has been addressed by many authors and several other distributions have been proposed, including the normal (Gaussian), log-normal and Gamma distributions.[2,28–31] The Weibull distribution cannot be favored with certainty as compared, for instance, with the Gaussian distribution, when a limited number of samples are subjected to the strength test.[29] The distribution may be changed, for instance, in non-homogeneous materials, such as composites and porous ceramics, due to the different mechanisms, e.g., residual stresses.

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When the amount of available experimental strength data is modest, it is usually impossible to state with certainty that the Weibull distribution is correct, and not, for instance, the Gaussian. Different methods offer some reliability factors, which enable a quantitative comparison of the successfulness when using different distributions for the same set of experimental data. An example is the correlation coefficient in the linear regression method, which measures the deviations of the data points from a straight line in the appropriate linearized dependence of the probability on the strength.

In statistics, $P-P$ plots ($P$ standing for the cumulative probability) or $Q-Q$ plots ($Q$ standing for the quantile) are often used to obtain a visual impression of how well the known theoretical distribution fits the experimental data.[32,33] A good match of plot points to the 45° line indicates a good agreement between the experiment and the theory for both types of plots (the physical units on both axes correspond to the probabilities in the case of the *P–P* plots, or the measured quantity in the case of the *Q–Q* plots).

In our previous paper we analyzed a large quantity of Monte–Carlo data and estimated the Weibull parameters using the ML method.[34] We combined theoretical results with the results of our measurements of the four-point bend strength of 96% alumina samples from a serial production (1000 strength values). We focused mostly on the problem of the reliability of the estimation of the Weibull modulus for a small number of samples. In particular, we confirmed the log-normal distribution of the estimated values of the Weibull modulus when a large set of data is randomly divided into small subsets.

In this work we make a statistical evaluation of 5100 experimental values of the bend strength of test pieces from a serial production of alumina products. We use the ML method for an estimation of the statistical parameters together with a *Q–Q*-plot to show that the Weibull distribution best fits the experimental data.

## 2. Experimental

Ceramic samples were fabricated using the low-pressure injection-molding technique in the company Hidria AET d.o.o. for quality-control purposes. The strengths of 5100 samples in the shape of a rectangular bar with dimensions of 4 mm × 3 mm × 45 mm, collected from 425 batches, were used in the study (there were 12 broken test pieces in each batch). The material was high alumina ceramic with a density of 0.95 of the theoretical value. The ceramic was prepared by sintering for 3 h at 1640 °C. The feedstock for injection molding was made from a powder containing 96% alumina ($d_{10} = 0.7$ μm, $d_{50} = 1.9$ μm, $d_{10} = 4.2$ μm) and 4% silica-based material ($d_{10} = 0.7$ μm, $d_{50} = 4.8$ μm, $d_{90} = 9.5$ μm), which served as a liquid-phase sintering aid. The numbers in brackets correspond to the particle diameters, where the cumulative size distribution reaches values of 10%, 50% and 90%, respectively. The material is primarily used for electrical insulating purposes and not high-strength-demanding tasks and is labeled as a "middle-strength" alumina ceramic in the company.

The strength was calculated from the breaking force in a 4-point bending test[35] using the equation:

$$\sigma = \frac{3F(L_S - L_L)}{2ah^2} \quad (1)$$

where $\sigma$ is the bending strength, $F$ is the breaking force, $L_S = 40$ mm is the outer support span, $L_L = 20$ mm is the load span, $a = 4$ mm is the specimen width, and $h = 3$ mm is the specimen thickness.

## 3. Statistical model and graphical representation

Our statistical variable is the four-point bend strength (called strength for brevity), denoted by the symbol $\sigma$. In our calculations we deal with both probability distribution functions: the probability density function $p(\sigma)$, and the cumulative probability function, also called the unreliability function, which is defined as: $P(\sigma) = \int_0^{\sigma} p(x) \ dx$. We test the statistical compatibility of the experimental data with the four different 2-parametric distribution functions: (1) Weibull, (2) normal (Gaussian), (3) log-normal, and (4) Gamma. The exact mathematical formulae are described in Section 3.2.

### *3.1. The procedure to estimate the goodness of fit*

The goodness of fit for a specific distribution was estimated from probability plots, where the experimental data is plotted against values calculated with a theoretical distribution. This is a graphical technique for assessing how well a certain distribution can describe experimental data. The graphical method, where all the experimental data are plotted, gives an important qualitative estimation about how well a particular distribution describes the data. The visualization of all the strength data in the evaluation of the distribution reliability is more illustrative and trustworthy than merely giving a number that indicates the level of correspondence of the theoretical distribution to the real experimental data.

The detailed procedure for constructing probability plots for each considered statistical distribution consists of the following steps:

(a) The best fitting parameters are determined for each distribution by using the maximum-likelihood method. This is done in the following way. The ($N = 5100$) measured strength values, $\sigma_i$, $i = 1$ to $N$, are inserted into the probability density function $p(a,b;\sigma)$, where $a$ and $b$ stand for the corresponding free parameters of the distribution, e.g., $a \equiv m$ and $b \equiv \sigma_0$ for the Weibull distribution, etc. The ML procedure maximizes the following function with respect to the free parameters $a$ and $b$:

$$Y = \ln\left(\prod_{i=1}^{N} p(a,b;\sigma_i)\right) = \sum_{i=1}^{N} \ln \ p(a,b;\sigma_i) \quad (2)$$

by setting to zero the derivatives of $Y$ with respect to $a$ and $b$. The detailed procedure is different for each distribution. The equations that were used to calculate each parameter for