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# Universal power-law strengthening in metals?

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ABSTRACT

The strength of most metals scales with either an internal or external length scale. Motivated by the wide applicability of this phenomenon to material type and microstructure, we develop a model which gives quantitative insight into the scaling exponent using the known universal properties of a dislocation network and the leading order stress dependence of an underlying critical stress distribution. The approach is found to be equally valid for both Hall–Petch strengthening and the smaller-is-stronger paradigm of small scale plasticity.



One of the scientifically most studied problems in the field of strength of crystalline materials is strengthening via grain size-reduction. This was experimentally demonstrated in the 1950s by both Hall [1] and Petch [2] for mild steel, ingot iron, spectrographic iron, as well as Zn. The stress, $\sigma$, at which strength is measured scales for all of these polycrystalline metals as $\sigma = \sigma_0 + kd^{-n}$, where $\sigma_0$ is some base resistance of the constituting single crystal, $d$ is the grain size, and $k$ is commonly referred to as the Hall–Petch constant. The Hall–Petch exponent, $n$, is typically $\simeq 0.5$. This empirical and technologically relevant scaling between strength and grain size appears simple, but has remained a fundamental challenge in metal physics. In fact, numerous mechanistic models have been proposed (dislocation pile-up, work hardening, composite models, etc., see Ref. [3] and references therein) to explain this scaling.

The persistence of the power-law scaling above a certain critical size $d$ is well reflected by the fact that not only very different microstructures (well annealed versus heavily cold rolled Ni, well annealed Fe, pearlitic steel, martensitic steel, tempered steel, etc.) obey Hall–Petch strengthening, but so do also fundamentally different parameters such as the yield strength, the lower yield point, the maximum flow strength, and the hardness or (see Petch [2]) the cleavage strength at $-198\,°C$. Since the range of microstructures covers everything between low defect densities in large pure crystallites, and immensely complex hierarchical defect structures of advanced steels that include precipitates, carbides, various types of phases, grain and dislocation boundaries, lathe pockets and dislocation density gradients, it urges the question if any single mechanistic picture can be held responsible for this phenomenon?.

A power-law strengthening with respect to a micro structural length scale is also seen in dynamic recrystallization [4] and recovery [5]. Indeed, for the case of recrystallization, Derby [4] has demonstrated power law scaling for a wide range of materials including different grades of steels, Cu, Ni, Mg, Fe, FeS, and also the non-metals NaCl, $NaNO_3$, olivine and ice, with the exponent $n$ ranging between 0.5 and 0.8. This has also been shown for ultra fine grade metals with approximately $100 < d < 3000$ nm tested between $-196°C$ and $720\,°C$ [6]. A third prominent example of power-law scaling is the "smaller is stronger" paradigm of micron- and nano-sized single crystals [7,8]. For this extrinsic size effect, where $d$ characterizes an external length scale, $n$ typically covers values between 0.2 and 0.7, and is for example found to depend on the initial dislocation density [9].

The above motivates Fig. 1, which summarizes literature data for Hall–Petch strengthening [1,2], dynamic recrystallization [4], and size-affected strength (see for example Ref. [10] and references therein). Fig. 1 demonstrates the remarkable fact that strength follows a similar power law with respect to both intrinsic (internal) and extrinsic (external) length scales for a vast range of materials and microstructure.

In this letter we extend previous work [10] rationalising the "smaller is stronger" paradigm as a general statistical sampling effect, to the much broader phenomenon of grain size strengthening and the Hall–Petch relation. The approach requires no specific

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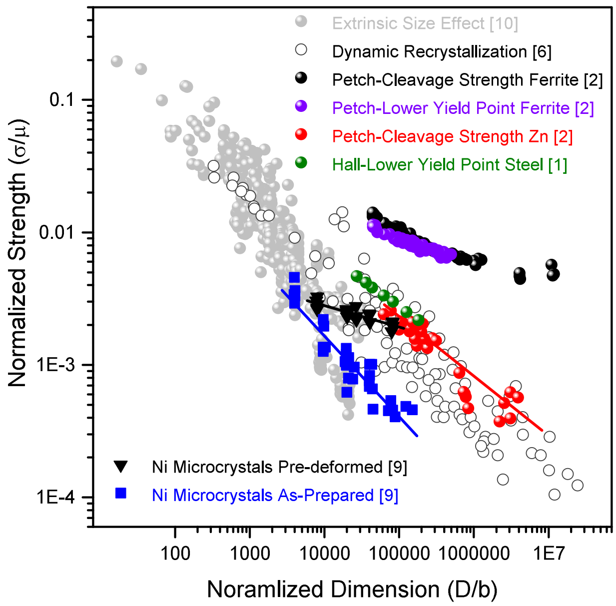


**Fig. 1.** Log–Log plot of strength versus an internal or external length scale for a wide range of literature data for both small scale plasticity and grain size data, including the original data from both Hall and Petch. Following Derby [4], the shear-strength versus length scale data is plotted in the respective units of an appropriate shear modulus and Burgers vector magnitude.

mechanism (although none is discounted) and originates from only a knowledge that a dislocation network exhibits scale-free behavior and that the extreme value statistics of a critical stress distribution is at play. By doing so, it is found that grain size strengthening, vis á vis the Hall–Petch mechanism, and the "smaller is stronger" paradigm, can be rationalized based on the same approach. This result also gives quantitative insight into the extent to which the scaling in strength is a truly universal phenomenon.

Ref. [10] demonstrated that two quite different statistical effects contribute to the size effect in small scale plasticity, one occurring in stress and one in plastic strain. In what follows, only leading order algebraic trends are considered, an approach entirely compatible with the notion that logarithmic accuracy is sufficient for the emergence and identification of the size effect phenomenon in data-sets such as that shown in fig. 1. The general approach taken begins with the construct that a typical deformation sequence admits $M$ discrete plastic events and that the stress and plastic strain value of each is sampled independently from a distribution of critical stresses and a distribution of plastic strain magnitudes. The current work then deals with the average with respect to many realizations of the deformation sequence, where in the case of the critical stress distribution the average ascending order is important and is determined via extreme-value-statistics. For the plastic strain sequence the order is assumed not to be so important and the average results in $M$ plastic strains all scaling with the mean of the plastic strain magnitude distribution. For the micro-plastic regime of deformation such a simplified picture should capture the leading order dependencies of the average stress–strain curve.

For the stress scaling, the internal dislocation network is characterized by a positive valued distribution, $P(\sigma)$, of critical stresses. Each such critical stress is the stress required for an irreversible rearrangement of the dislocation network and thus a plastic event. Such a plastic event is typical of intermittent plasticity and is generally referred to as a dislocation avalanche, which my be characterized in terms of the plastic strain it admits and the energy it releases [11–13]. Here only the former, plastic strain, will be considered. For a given elemental volume, $L^3$, there exists $M = \rho L^3$ such critical stresses ($\rho$ being the density of the available critical stresses). Sampling the distribution $M$ times gives a sequence of critical stresses, the smallest of which play the dominant role in initiating the transition to plastic flow. If $M$ is large then the statistics of the extreme controls these relevant critical stresses, whereas if $M$ is small then the statistics of the most probable becomes relevant. This rather general description naturally results in a shift to higher critical stresses when volume (and therefore $M$) decreases.

For sufficiently large $M$ ($M > 100$), the apparatus of extreme value statistics defines the characteristic $i$th critical stress, $\sigma_i$, of the ordered sequence via [10]

$$i = M \int_0^{\sigma_i} \mathrm{d}\sigma\, P[\sigma]. \tag{1}$$

The above is a generalization of the well known $i = 1$ case of the average minimum value of a sampled ordered sequence of size $M$ [14]. For the small-stress regime, $P[\sigma] \sim \sigma^{\alpha}$, and Eq. (1) leads to $\sigma_i \sim (i/M)^{1/(1+\alpha)} \sim (i/L^3)^{1/(1+\alpha)}$. Thus, as the volume reduces the stress scale increases. Apart from $\alpha$, this result is independent of the overall form of the positive valued distribution.

For strain scaling, the universal finite-size scaling properties of a dislocation network in a state of criticality is exploited [11]. In particular, like that of earth quakes, avalanche sizes and crackling noise (see for example Refs. [12,13]), the distribution of strain magnitudes, $\delta\varepsilon$, associated with intermittent plasticity follows a power-law form with a non-algebraic scaling function (prefactor), $f[\cdot]$. Here $f[\cdot]$ depends on a length scale which in Ref. [10] characterized the sample volume. That is, $P[\delta\varepsilon] \sim f[\delta\varepsilon/\delta\varepsilon_{\max}(L)]\delta\varepsilon^{-\tau}$ where $\tau$ is a universal scaling exponent for intermittent plastic strain activity [11,15] and $\delta\varepsilon_{\max}(L)$ varies inversely with $L$ [15,16]. Thus the plastic strain magnitude scale will be characterized by some function of $L$. Using a well-accepted representation of the scaling function [10], this characteristic scaling is found to be $\delta\varepsilon_i \sim L^{\tau-2}$, which gives the simple scaling of total plastic strain at the $i$th plastic event as $\varepsilon_i \sim iL^{\tau-2}$. We note that the prefactor of this relation depends on both the minimum plastic strain and $\delta\varepsilon_{\max}(L)$, however the resulting non-algebraic dependence on $L$ does not affect the leading order algebraic contribution to the size effect.

When put together, $\sigma_i \sim (i/L^3)^{1/(1+\alpha)} \sim (\delta\varepsilon_i L^{2-\tau}/L^3)^{1/(1+\alpha)}$, and the critical stress at a fixed plastic strain is found to scale as $L^{-(\tau+1)/(\alpha+1)}$ giving a size effect exponent of $n = (\tau+1)/(\alpha+1)$. A more extended derivation of this result may be found in Ref. [10] and recent experimental verification of some aspects of this prediction may be found in Ref. [17].

The above approach can be generalized to a polycrystalline material in a straight forward manner by considering an ensemble of grains, whose characteristic volume is defined as $L^3_{\mathrm{grain}} = d^3$. This also defines $M_{\mathrm{grain}} = \rho L^3_{\mathrm{grain}}$. The critical stresses available to the bulk material are described by a single effective distribution where the total number of critical stresses available is given by $M_{\mathrm{bulk}} = \rho L^3_{\mathrm{bulk}}$. Eq. (1) then gives the $i$th average critical stress of the bulk polycrystalline system as $\sigma_i \sim (i/M_{\mathrm{bulk}})^{1/(1+\alpha)}$.

To see how a single effective distribution of critical stresses may represent the extreme value statistics of critical stresses of a polycrystalline environment, Eq. (1) is generalized to

$$i = \sum_{n \in \mathrm{grains}} M_n \int_0^{\sigma_i} \mathrm{d}\sigma\, P_n[\sigma], \tag{2}$$

where the $n$th grain is characterized by its own critical stress distribution $P_n[\sigma]$ and $M_n$. $P_n[\sigma]$ is expected to depend on grain shape, the local grain network structure and also grain orientation with