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Finite clusters of fast-rotating spinless bosons in a harmonic trap

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ABSTRACT

Rapidly rotating two-dimensional ultracold Bose–Einstein condensates of spinless bosons in a harmonic trap have attracted considerable interest during the recent years. It is expected that, in the fast-rotation limit, the system of bosons will exhibit collective behavior similar to that of two-dimensional electrons in the fractional quantum Hall effect regime. It is predicted that the most robust correlated bosonic state in this regime will be the Bose Laughlin state at a half filling factor. An exact treatment of such a state is generally a formidable task due to the inherent many-particle nature of the wave function. We report in this work that a transformation to Jacobi coordinates allows one to obtain much desirable exact analytic closed-form expressions for various quantities of interest corresponding to a Bose Laughlin wave function for various finite systems of particles.

1. Introduction

The integer quantum Hall effect (IQHE) [1–5] and the fractional quantum Hall effect (FQHE) [6–14] are two remarkable phenomena which occur in a two-dimensional electron gas (2DEG) subject to a perpendicular magnetic field. At first look, a rapidly rotating, dilute Bose–Einstein condensate (BEC) system made up of neutral atoms at very low temperature [15–24] seems to be unrelated to the physics of quantum Hall effect (QHE). Despite such an appearance, it is now fairly well understood that there exists a very close relationship between these two phenomena. This relationship originates from the fact that the role played by a perpendicular magnetic field in a 2DEG closely mirrors the effect created by fast rotation in BEC systems of harmonically confined two-dimensional (2D) bosons [25].

It has been predicted that for fast rotation, when the rotation frequency is close to the harmonic trap frequency, the BEC system will enter the QHE regime [26–35]. It is expected that, in this regime, the bosons will stabilize in strongly correlated liquid states that resemble the FQHE liquid states of a 2DEG of Coulomb-interacting electrons in a perpendicular magnetic field. If this is the case, one can expect the atomic system to exhibit most of the characteristic features of FQHE liquid states such as fractional elementary excitations and anyon statistics.

From this perspective, cold atomic systems offer a fascinating possibility in exploring the properties of strongly correlated systems in the regime where inter-particle interactions play a leading role. Cold atomic systems with enhanced inter-particle correlations have been the subject of many theoretical and experimental studies since these efforts may lead to the observation of novel physical phenomena. What is even more tempting is the fact that, since cold atomic gases can be well controlled and manipulated, they may provide us with new scenarios not yet available in the realm of other condensed matter systems. The challenging experimental task here is how to reliably stabilize a cold atomic system into the FQHE regime of strong correlations. In addition to being interesting in its own right, the prospect of producing the bosonic version of FQHE states in cold atomic systems may have other important ramifications, for instance, in quantum computing [36]. In recent years, the QHE-rotating BEC analogy has been theoretically explored in great detail, starting with the prediction of a Bose Laughlin state occurring in systems of fast-rotating spinless bosons in a harmonic trap [37,38].

In this work, we report exact analytical results for finite systems of particles described by the Bose Laughlin wave function at filling factor $\nu = 1/2$. We were able to calculate exactly the energy and one-particle density function corresponding to various systems with up to $N=4$ particles by transforming the desired quantities in terms of the so-called Jacobi coordinates. The method introduced is general and, thus, can be extended to larger systems of particles. The exact results so obtained can also serve as an important benchmark for numerical and exact diagonalization studies that routinely are limited to finite systems of few bosonic particles for instance $N=4$ or $N=5$ particles [39–41].

2. Analogy between a charged particle in a magnetic field and a rotating boson in a harmonic trap

Let us consider a particle with charge, q (for simplicity we may consider it to be positive), and mass, m, confined in a 2D space and subject to a perpendicular magnetic field in the z-direction,

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$\vec{B}=(0,0,B)$. The quantum Hamiltonian for such a particle is written as

$$\hat{H}=\frac{1}{2m}[\hat{\vec{p}}-q\vec{A}(\vec{r})]^2, \tag{1}$$

where $\hat{\vec{p}}=(\hat{p}_x,\hat{p}_y)$ is the 2D linear momentum operator, $\vec{A}(\vec{r})$ is the vector potential and $\vec{r}=(x,y)$ is the 2D position vector. Note that $\vec{B}=\vec{\nabla}\times\vec{A}(\vec{r})$. If one adopts the symmetric gauge

$$\vec{A}(\vec{r})=\frac{B}{2}(-y,x,0), \tag{2}$$

one can write the quantum Hamiltonian above as

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{m}{2}\omega^2r^2-\omega\hat{L}_z, \tag{3}$$

where $\omega=\omega_c/2$, $\omega_c=(qB)/m$ is the cyclotron frequency, $\hat{p}^2=\hat{p}_x^2+\hat{p}_y^2$, $r^2=x^2+y^2$, and $\hat{L}_z=x\hat{p}_y-y\hat{p}_x$ is the angular momentum operator in the z-direction.

Let us now consider a spinless boson particle with mass m in a 2D harmonic trap of strength ω, rotating with an angular frequency, Ω. In a rotating reference frame, the quantum Hamiltonian of the harmonically confined boson can be written as

$$\hat{H}_b=\frac{\hat{p}^2}{2m}+\frac{m}{2}\omega^2r^2-\Omega\hat{L}_z. \tag{4}$$

One can immediately see that the bosonic Hamiltonian $\hat{H}_b$ of a rotating system of harmonically confined bosons is basically equivalent to the Hamiltonian, $\hat{H}$, of charged particles in a magnetic field when $\Omega=\omega$. This observation suggests a close analogy between the BEC and the QHE phenomenon. By mapping

$$2m\omega\rightarrow qB, \tag{5}$$

one can write $\hat{H}_b$ for $\Omega=\omega$ exactly as in Eq. (1) where $q\vec{A}(\vec{r})$ is formally replaced with an effective vector potential, $\vec{A}_{eff}(\vec{r})$, as given below:

$$q\vec{A}(\vec{r})\rightarrow\vec{A}_{eff}(\vec{r})=m\omega(-y,x,0), \tag{6}$$

where now $q\vec{B}\rightarrow\vec{B}_{eff}=\vec{\nabla}\times\vec{A}_{eff}(\vec{r})$.

Hence, by formally mapping the bosonic problem to that of a QHE system, one can make use of well-known results pertaining to such systems. For example, one knows that the quantum mechanical solution of the one-particle QHE problem results in an energy spectrum of the form, $E_n=(n+1/2)\hbar\omega_c$ $(\omega_c=2\omega)$, where $n=0,1,\ldots$ is the quantum index of Landau levels (LLs). Each LL is highly degenerate with N_s single-particle states having the same energy. When one is interested in the lowest energy states, one focuses the attention on the lowest Landau level (LLL) that corresponds to $n=0$. As in the FQHE case, the basic non-trivial correlated state for a system of bosons would be the Bose Laughling state [42] at filling factor $\nu=1/2$ written as

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j)^2\,e^{-\sum_{j=1}^{N}|z_j|^2/4l_0^2}, \tag{7}$$

where $z_j=x_j+iy_j=r_je^{i\varphi_j}$ are the 2D position vectors in complex notation $(i=\sqrt{-1})$ and $l_0=\sqrt{\hbar/(qB)}$ is the characteristic length.

3. Bose Laughlin wave function and Jacobi coordinates

By adopting a QHE terminology, let us consider a 2D system of $N(\geq 2)$ particles with charge q and mass m subject to a strong perpendicular uniform magnetic field, B in the z-direction. The particles are spinless (or, as in the case of electrons in the FQHE regime, their spin is considered fully polarized in the direction of the magnetic field). The density of the system can be written as

$$\rho_0=\frac{\nu}{2\pi l_0^2}, \tag{8}$$

where the filling factor, ν, is defined as the ratio of N relative to N_s (the degeneracy of each LL). In a disk geometry, we may visualize the particles as filling uniformly a disk region with area, $\pi R_N^2=N/\rho_0$, where R_N is the radius of such a disk. The quantum Hamiltonian for an interacting system of particles, $\hat{H}=\hat{K}+\hat{V}$, consists of kinetic and potential energy operators where $\hat{K}=\sum_{i=1}^{N}[\hat{\vec{p}}_i-q\vec{A}(\vec{r}_i)]^2/(2m)$ is the kinetic energy operator. If we focus our attention only on the particle–particle interaction energy, one can write

$$\hat{V}_{pp}=\sum_{i<j}^{N}v(\vec{r}_i-\vec{r}_j). \tag{9}$$

To have the present results apply to real FQHE states, we consider the interaction potential between particles to be of Coulomb form, $v(\vec{r}_i-\vec{r}_j)=q^2/|\vec{r}_i-\vec{r}_j|$. Note that, for the sake of simplicity, the Coulomb's constant is omitted in the expression of the Coulomb interaction potential. Other interaction potentials can be equally handled with ease as long as they are translationally invariant.

Because the Bose Laughlin wave function includes only states in the LLL, the expectation value of the kinetic energy operator is a mere constant, $\langle\hat{K}\rangle/N=\frac{1}{2}\hbar\omega_c$ where $\hbar$ is reduced Planck's constant and ω_c is the cyclotron frequency. The most important ingredient entering various expressions for the particle–particle interaction energy or similar quantities is the magnitude square of the wave function. For the Bose Laughlin wave function, we can write it as

$$|\Psi(z_1,\ldots,z_N)|^2=[F_N(\vec{r}_1,\ldots,\vec{r}_N)]^2\exp\left[-\frac{S_N(\vec{r}_1,\ldots,\vec{r}_N)}{2l_0^2}\right], \tag{10}$$

where

$$F_N(\vec{r}_1,\ldots,\vec{r}_N)=\prod_{i<j}^{N}(\vec{r}_i-\vec{r}_j)^2, \tag{11}$$

and

$$S_N(\vec{r}_1,\ldots,\vec{r}_N)=\sum_{j=1}^{N}|\vec{r}_j|^2. \tag{12}$$

At this juncture, we make the observation that various expressions that involve $F_N(\vec{r}_1,\ldots,\vec{r}_N)$ and $S_N(\vec{r}_1,\ldots,\vec{r}_N)$ are more conveniently written in terms of Jacobi coordinates. For the sake of brevity, the reader is referred to Appendix A for a description of Jacobi coordinates and additional details. An excellent and concise introduction to Jacobi's coordinates is also provided in [43] (see pp. 423 and 608).

For a given many-particle wave function, the interaction energy per particle is written as

$$\epsilon_{pp}=\frac{\langle\hat{V}_{pp}\rangle}{N}=\frac{(N-1)}{2}\langle v(\vec{r}_1-\vec{r}_2)\rangle, \tag{13}$$

where in a short-hand notation $\langle\cdots\rangle$ represents the standard quantum expectation value of the given operator with respect to the wave function. When calculating $\langle\hat{V}_{pp}\rangle$ from Eq. (13) one clearly sees that the interaction potential, $v(\vec{r}_1-\vec{r}_2)$, depends only on the magnitude of the first Jacobi coordinate, $\xi_1=|\vec{\xi}_1|$. This implies that rewriting $F_N(\vec{r}_1,\ldots,\vec{r}_N)$ and $S_N(\vec{r}_1,\ldots,\vec{r}_N)$ in terms of the Jacobi coordinates will significantly simplify many calculations that otherwise are very challenging because of the inherent many-particle nature of the wave function.

This approach enabled us to obtain simple expressions for the functions $F_N(\vec{r}_1,\ldots,\vec{r}_N)$ and $S_N(\vec{r}_1,\ldots,\vec{r}_N)$ in terms of Jacobi