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Modeling of active and passive nonlinear metamaterials

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Abstract

We develop general results for nonlinear metamaterials based on simple circuit models that reflect the elementary nonlinear behavior of the medium. In particular, we consider both active and passive nonlinearities which can lead to gain, harmonic generation and a variety of nonlinear waves depending on circuit parameters and signal amplitude. We show that the medium can exhibit a phase transition to a synchronized state and derive conditions for the transformation based on a widely used multiple time scale approach that leads to the well-known Complex Ginzburg–Landau equation. Further, we examine the variety of nonlinear waves that can exist in such systems, and we present numerical results for both active and passive metamaterial cases.

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1. Introduction

The potential of engineered materials that possess novel dielectric and magnetic properties is well known and has been intensively studied in recent years. Much attention has been focused on the properties associated with two- or three-dimensional arrays of subwavelength unit cells comprised of electromagnetic resonators. These so-called metamaterials represent a burgeoning and valuable research area [1–6]. While the original metamaterial concept was based on the linear dispersive properties of unit cells, it was also recognized that nonlinear metamaterials could greatly enhance certain properties [4]. Pioneering models were developed that were based on the introduction of nonlinear circuit elements into the unit cells [7,9,8,10] which were shown to exhibit a rich diversity of behavior, including fascinating phenomena dependent on the so-called left-handed properties of metamaterials. These inspired further development in the following Refs. [11–18]. Indeed, the original inspiration for nonlinear metamaterials may have come from Ref. [19] where it was first shown that phase conjugation could be realized with the use of a nonlinear magnetic medium constructed from an ensemble of nonlinear elementary oscillators. The more recent generalization of similar ideas was borne out in studies of nonlinear transmission lines [20–24], which have been found to display a similarly wide variety of frequency and amplitude dependent phenomena.

Moreover, an array of coupled nonlinear oscillators was the basis of the famous Fermi, Pasta, Ulam

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system [25,26] that was first studied numerically and found to contain unexpected effects, namely a repeated recurrence of a given initial perturbation, and persistent frequency mixing phenomena. This observation is widely acknowledged to have inspired the modern development of solitons and the very powerful analytical methods for delineating their properties.

In general we may divide nonlinear metamaterial arrays into two categories: active and passive. In the active case, the cells contain elements that impart gain to the medium and a source of free energy exists. The existence of gain is akin to that of a lasing medium [27]. In this context, very simple nonlinear resonator circuits were shown to exhibit frequency mixing and chaos as isolated cells [28], however it is not clear how these complex phenomena might manifest themselves in an array.

On the other hand, a comprehensive study of one, two and three-dimensional arrays of metamaterial cells has been undertaken in the linear case by Shamonina et al. [29]. They showed that nearest-neighbor interactions are sufficient to give rise to a class of magnetoinductive waves whose dispersion relations are similar to those of waves propagating in a crystalline lattice. They explored the geometric properties of intercell coupling and implications this coupling has on the anisotropy of the medium. Of particular interest in their work is the development of dispersion relations which describe the wave propagation in the limit where the unit cell dimensions are much smaller than a wavelength, also considered in Refs. [7,8]. From this work, a key question arises as to how it might be possible to incorporate the complex nonlinear behavior that can occur in a single cell into the dispersion relation description of an array.

It is this question that has compelled us and others [50] to study nonlinear metamaterial arrays in particular, and we wish to clarify the dynamical regimes that manifest themselves in such systems. In particular, we are interested in the frequency agility and gain characteristics of nonlinear metamaterials, and in identifying the forms of nonlinear waves that may exist. We will build upon the nearest-neighbor interaction model of Ref. [29] and treat examples of both active and passive nonlinearities. However, it is also important to note, as shown in Ref. [29], that nearest-neighbor interactions are at best a crude approximation in some cases, particularly in extending the models to two and three dimensions. We will nonetheless restrict ourselves to the lowest-order spatial coupling in order to demonstrate the basic concepts, justified in part because our analysis methods can readily be applied to more complex coupling situations. In this work the specific active nonlinearity is introduced by the insertion of a biased tunnel diode into the resonator circuit. Whereas in the passive category, any nonlinear element which alters its properties with signal amplitude can be considered, and we have chosen to explore the often-used varactor. However, the theoretical methods we apply are general and can be used for virtually any nonlinearity of interest.

In this work, we wish to take advantage of the fact that an array of coupled nonlinear oscillators is essentially the heart of many physical systems of importance in physics and engineering. Plasmas, lasers, superconductors, liquid crystals, even Bose–Einstein condensates among many others, all share characteristics that arise from coupled oscillator arrays. As such, many of the mathematical techniques developed in these fields can find application in the development of viable models for nonlinear metamaterials. In particular, we recognize that all these physical systems generally contain two very disparate time scales: a fast time scale over which the phase of an element may vary, and a slower time scale over which adjacent cells may exchange energy. By appropriately averaging the fast time scale, we are led to the celebrated Complex Ginzburg–Landau equation (CGLE) [30], developed to explain phase transitions in superconductors. A related set of equations will be developed for our case, and some of its many diverse properties will be found to describe the behavior of these metamaterials.

We note that the concept of phase transitions in metamaterials has been introduced by the work of Zharov et al. [9]. They described the behavior of nonlinear permeability and permittivity, which effectively leads to a nonlinear frequency shift and gives rise to interesting hysteresis phenomena. Extensions of this concept to include nearest-neighbor interactions leads to the existence of breather solitons, which have been explored in Refs. [31–33]. This work was focused on passive nonlinearities associated with nonlinear reactances and the nonlinear waves studied in these papers relied upon the condition of weak or nonexistent dissipation. However, only a few nonlinear dissipative systems are known to be integrable [34], and not all the existing solutions for a given situation may have been identified. In this work, we wish to develop analytical methods that will allow us to describe general metamaterial device configurations, and classify all relevant nonlinear behavior. Our aim in this paper is thus to extend these concepts in a general way to both active and passive media. Specifically, we shall be interested in both coherent and incoherent, i.e. chaotic, behavior that may be associated with arrays of nonlinear cells. In a manner that parallels the development of the understanding of the Fermi, Pasta