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Quantum mechanical treatment of traveling light in an absorptive medium of two-level systems

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ABSTRACT

Quantum mechanical treatment of a light wave that propagates through an absorptive medium is presented. Unlike a phenomenological beam-splitter model conventionally employed to describe a traveling light in a lossy medium, the time evolution of the field operator is derived using the Heisenberg equation with the Hamiltonian for a physical system, where the light wave interacts with an ensemble of two-level systems in a medium. Using the obtained time-evolved field operators, the mean values and variances of the light amplitude and the photon number are evaluated. The results are in agreement with those obtained in the beam-splitter model, giving a logical theoretical basis for the phenomenological beam-splitter model.

1. Introduction

Quantum fluctuations or quantum noises are of fundamental interest in quantum optics, including inherent fluctuations in coherent states and squeezed states, vacuum fluctuations, and quantum-limited noise figures in phase-insensitive or phase-sensitive optical amplifiers. It is well known that quantum properties of a light wave are affected by propagating loss in a medium. Conventionally, traveling light in a dielectric medium was quantum mechanically treated with a quantum mechanical version of the Maxwell's equations that includes a phenomenological noise field operator or an operator phenomenologically representing an absorption phenomenon [1–7]. A spatial differential equation of the light field operator (annihilation operator) was derived from the quantum mechanical Maxwell's equations, by which quantum properties of a traveling light was analyzed. A beam-splitter model was also suggested similar to the spatial differential equation [8–15], in which a noise field operator is assumed to be overlapped onto the attenuated light field operator via a beam splitter representing a loss phenomenon. This beam splitter model has been widely utilized in considering quantum properties of light wave propagating in a lossy medium because of its simplicity.

In a beam-splitter model, the in-out relationship of the light field operator through a loss medium is expressed as [8,15]

$$\hat{a}_{out} = \tilde{t}\hat{a}_{in} + \tilde{r}\hat{a}_{vac} \quad (1)$$

with $|\tilde{t}|^2 + |\tilde{r}|^2 = 1$, where $\hat{a}_{in}$ and $\hat{a}_{out}$ are the field operators at the input and output of the medium, respectively, $\hat{a}_{vac}$ is the vacuum field (or noise field) operator, and $\tilde{t}$ and $\tilde{r}$ are the amplitude transmittance and reflectance of a beam splitter, respectively. The first term in Eq. (1) represents the attenuation of the incident light, and the second term represents a noise field overlapped with the incident light caused by a reaction of some loss mechanism. While this beam-splitter model is convenient and useful, it was phenomenologically presented, not directly derived from the first principles of quantum mechanics. To the best of the author's knowledge, a logical derivation from the fundamentals of the quantum mechanical theory has not been reported.

Based on the above background, this paper presents a quantum mechanical description of a light wave passing through an absorptive medium. Interactions between light and an ensemble of two-level systems are assumed to cause attenuation of a traveling light, and the space evolution of the light wave state is derived using the Heisenberg equation with the Hamiltonian for such a physical system. The results are in agreement with those obtained in the beam-splitter model, thus providing a theoretical justification of the phenomenological beam-splitter model.

2. Theoretical treatment

2.1. Time evolution of the field operator

We consider an absorptive medium, in which traveling light is attenuated through interaction with an ensemble of two-level

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systems [16]. The Hamiltonian for such a physical system is expressed as [15]

$$\hat{H} = \hbar\omega\hat{a}^{\dagger}\hat{a} + \sum_j \hbar\omega_m^{(j)}\hat{\pi}_j^{\dagger}\hat{\pi}_j + i\sum_j \hbar(\alpha_j\hat{\pi}_j^{\dagger}\hat{a} - \alpha_j^*\hat{\pi}_j\hat{a}^{\dagger}). \tag{2}$$

The first, second, and third terms represent the Hamiltonians for the light field, an ensemble of two-level systems in the medium, and the interaction between the light and the two-level systems, respectively. Here, $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and creation operators of light, respectively; $\hat{\pi} = |2><1|$ and $\hat{\pi}^{\dagger} = |2><1|$ are the transition operators of a two-level system in the medium with $|2>$ and $|1>$ denoting the upper and lower energy levels, respectively; $\hbar$ is the reduced Planck's constant; ω is the lightwave angular frequency; $\hbar\omega_m$ is the energy difference between the two levels; α_j is a proportional constant; and the subscript j labels the two-level systems.

The time evolution of the field operator and the transition operators is governed by the Hamiltonian $\hat{H}$ of the composite system through the Heisenberg equation:

$$\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, \hat{H}] = -i\omega\hat{a} - \sum_j \alpha_j\hat{\pi}_j, \tag{3a}$$

$$\frac{d\hat{\pi}_j}{dt} = \frac{1}{i\hbar}\left[\hat{\pi}_j, \hat{H}\right] = -i\omega_m^{(j)}\hat{\pi}_j + \alpha_j(\hat{\pi}_j\hat{\pi}_j^{\dagger} - \hat{\pi}_j^{\dagger}\hat{\pi}_j)\hat{a}. \tag{3b}$$

These equations can be simplified by rewriting the operators as $\hat{a} \to \hat{a}(t)e^{-i\omega t}$ and $\hat{\pi}_j \to \hat{\pi}_j(t)e^{-i\omega_j t}$:

$$\frac{d\hat{a}}{dt} = -\sum_j \alpha_j^*\hat{\pi}_j e^{-i(\omega_j-\omega)t}, \tag{4a}$$

$$\frac{d\hat{\pi}_j}{dt} = \alpha_j(\hat{\pi}_j\hat{\pi}_j^{\dagger} - \hat{\pi}_j^{\dagger}\hat{\pi}_j)\hat{a}e^{i(\omega_j-\omega)t} = \alpha_j\hat{\Pi}_j\hat{a}e^{i(\omega_j-\omega)t}, \tag{4b}$$

where $\hat{\Pi}_j \equiv \hat{\pi}_j\hat{\pi}_j^{\dagger} - \hat{\pi}_j^{\dagger}\hat{\pi}_j$ is a shorthand notation.

We solve Eqs. (4) by employing an iterative approximation. First, the first-order solutions are obtained by substituting the initial values $\{\hat{a}^{(0)}, \hat{\pi}_j^{(0)}\}$ into the right-hand side of Eqs. (4):

$$\frac{d\hat{a}}{dt} = -\sum_j \alpha_j^*\hat{\pi}_j^{(0)} e^{-i(\omega_j-\omega)t}, \tag{5a}$$

$$\frac{d\hat{\pi}_j}{dt} = \alpha_j\hat{\Pi}_j^{(0)}\hat{a}^{(0)} e^{i(\omega_j-\omega)t}. \tag{5b}$$

The solutions of these equations are

$$\hat{a}(t) = \hat{a}^{(0)} - i\sum_j \alpha_j^*\frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}, \tag{6a}$$

$$\hat{\pi}(t) = \hat{\pi}_j^{(0)} - i\alpha_j\frac{e^{i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{a}^{(0)}\hat{\Pi}_j^{(0)}. \tag{6b}$$

Next, we substitute these first-order solutions into the right-hand side of Eq. (4a):

$$\begin{aligned}\frac{d\hat{a}}{dt} &= -\sum_j \alpha_j^*\left[\hat{\pi}_j^{(0)} - i\alpha_j\frac{e^{i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{a}^{(0)}\hat{\Pi}_j^{(0)}\right]e^{-i(\omega_j-\omega)t} \\ &= -\sum_j \alpha_j^*\hat{\pi}_j^{(0)}e^{-i(\omega_j-\omega)t} \\ &\quad - i\hat{a}^{(0)}\sum_j \left|\alpha_j\right|^2\frac{1}{\omega_j - \omega}\left\{e^{-i(\omega_j-\omega)t} - 1\right\}\hat{\Pi}_j^{(0)}.\end{aligned} \tag{7}$$

The solution of this equation is given by

$$\begin{aligned}\hat{a}(t) &= \hat{a}^{(0)}\left[1 - \sum_j \left|\alpha_j\right|^2\frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j - \omega)t}{(\omega_j - \omega)^2}\hat{\Pi}_j^{(0)}\right] \\ &\quad - i\sum_j \alpha_j^*\frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}.\end{aligned} \tag{8}$$

We assume the interaction time is short and consider the above expression of $\hat{a}(t)$ as the solution of Eq. (4a).

2.2. Physical quantities

Physical quantities of a light wave are expressed in terms of expectation values of $\hat{a}(t)$ in an initial state of the composite system under consideration. The mean amplitude is given by $<a> = <\Psi_0|\hat{a}(t)|\Psi_0>$, where $|\Psi_0>$ is an initial state of the composite system of light and medium. Here, we are considering an absorptive medium, not an amplifying one, and thus, we assume that all two-level systems are initially in the lower energy states. Such an initial state can be expressed as

$$\left|\Psi_0\right> = \left|\Psi_r^{(0)}\right> \otimes \left|\Psi_m^{(0)}\right> \tag{9}$$

with

$$|\Psi_m^{(0)}> = \bigotimes_j |1>_j, \tag{10}$$

where $|\Psi_r^{(0)}>$ and $|\Psi_m^{(0)}>$ denote the initial states of the light and the medium, respectively. Applying this initial state to the time-evolved field operator $\hat{a}(t)$ given by Eq. (8), we find that the mean amplitude of the light wave at time t is

$$<a(t)> = <a(0)>\left\{1 - \sum_j |\alpha_j|^2\frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j - \omega)t}{(\omega_j - \omega)^2}\right\}, \tag{11}$$

where $<a(0)> \equiv <\Psi_r^{(0)}\left|\hat{a}^{(0)}\right|\Psi_r^{(0)}>$, and $<\Psi_m^{(0)}\left|\hat{\pi}_j^{(0)}\hat{\pi}_j^{(0)\dagger}\right|\Psi_m^{(0)}> = 1$, $<\Psi_m^{(0)}\left|\hat{\pi}_j^{(0)\dagger}\hat{\pi}_j^{(0)}\right|\Psi_m^{(0)}> = 0$, and $<\Psi_m^{(0)}\left|\hat{\pi}_j^{(0)}\right|\Psi_m^{(0)}> = 0$ have been used to obtain the result.

The second term in Eq. (11) includes information of the energy states in the medium, which can be simplified as follows. First, we decompose the second term into the real and imaginary parts as

$$\begin{aligned}&\sum_j |\alpha_j|^2\frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j - \omega)t}{(\omega_j - \omega)^2} \\ &= 2\sum_j |\alpha_j|^2\frac{\sin^2[(\omega_j - \omega)t/2]}{(\omega_j - \omega)^2} + i\sum_j |\alpha_j|^2\frac{\sin[(\omega_j - \omega)t] - (\omega_j - \omega)t}{(\omega_j - \omega)^2}.\end{aligned} \tag{12}$$

Under the condition that the energy states are densely distributed in the frequency domain, the summation in the real part can be replaced by an integral, i.e.,