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Continuous spectrum of modes for optical micro-sphere resonators

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ABSTRACT

This paper presents an improved modal analysis for the spherical dielectric resonator. This is commonly carried out by assuming an outgoing spherical Hankel function for the region surrounding the dielectric sphere. It will be shown that this assumption is incomplete and cannot lead to the entire spectrum of resonance frequencies. Following an analytical formulation, we prove that, like cylindrical resonators, the only choice for the outer region of the dielectric sphere can be a proper linear combination of an inward and an outward traveling wave. Starting from this formulation, we determine the complete spectrum of the resonance frequencies and the associated mode fields. In this analysis, the continuous spectrum of resonance frequencies is introduced and the properties of radiation modes are studied in detail. The proposed analytical formulation is thereafter employed to calculate the quality factor of the resonator due to radiation and dielectric loss.

1. Introduction

Optical dielectric resonators have found a number of applications in photonics such as in frequency comb generators, sensors, parametric amplifiers, and lasers, to name a few [1–3]. In comparison with many other resonators, spherical dielectric resonators possess large values of quality factor despite their relatively simple fabrication process [1]. Obviously, determination of the electric and magnetic field distributions for the resonator modes is the first step for acquiring the required insight into the operation principle of spherical dielectric resonators. So far, many researchers have studied such open resonators. In the existing analyses, the spherical Bessel functions have been used for the electromagnetic field representation both for the interior and exterior regions of the resonator [4–13]. In [4–6], the spherical Hankel function and in [7–11] the spherical Neumann function have been assumed for the outside region whereas in [12,13] exponential functions have been utilized. They mainly deal with certain resonator modes with discrete resonance frequencies.

In this work, starting from the characteristic equation for the spherical dielectric resonator, we show that a complete expression for the electromagnetic field in the surrounding region of the dielectric sphere must involve both an outgoing and an incoming spherical wave. This formulation leads to the complete set of modes which, because of the unbounded outside region, must also include a continuous spectrum of resonance frequencies.

Note that a continuous spectrum also exists in rectangular and cylindrical waveguides. For instance, in the dielectric slab waveguide of Table 1, apart from a limited number of discrete propagation constants (Table 1(a)), there exists a continuous spectrum of propagation constants k_z with $0 < k_z < k_0$. These modes can be assumed as being generated by two oblique plane waves impinging on the dielectric slab from above and beneath as shown in Table 1(b) [14]. In this way, the total active power perpendicular to the slab will remain zero and propagation along the z-direction follows $\exp(i\,k_z\,z)$. Likewise in the cylindrical waveguide shown in Table 1, beside the discrete modes (Table 1(c)) which are confined to the core, there is a continuous spectrum of modes (Table 1(d)) as a result of a superposition of $H_n^{(2)}(k_0 r)$ and $H_n^{(1)}(k_0 r)$ outside the core region [15,16]. On the other hand, it goes without saying that determination of propagation constant of waveguides is related to determination of resonance modes of resonators through the well-known method of transverse resonance. Hence, keeping the aforementioned properties of the continuous spectrum of rectangular and cylindrical waveguides in mind, we have to include both an outward and an inward traveling spherical wave (Table 1 (f)) to investigate the continuous spectrum of resonance modes of a spherical dielectric resonator. In this work, we study these resonance modes and explain how they can be distinguished from the modes belonging to the discrete spectrum. Furthermore, we study modes with near-zero resonance frequencies.

Another aspect we investigate in this work is the radiation loss of a spherical dielectric resonator and its relation to the dielectric

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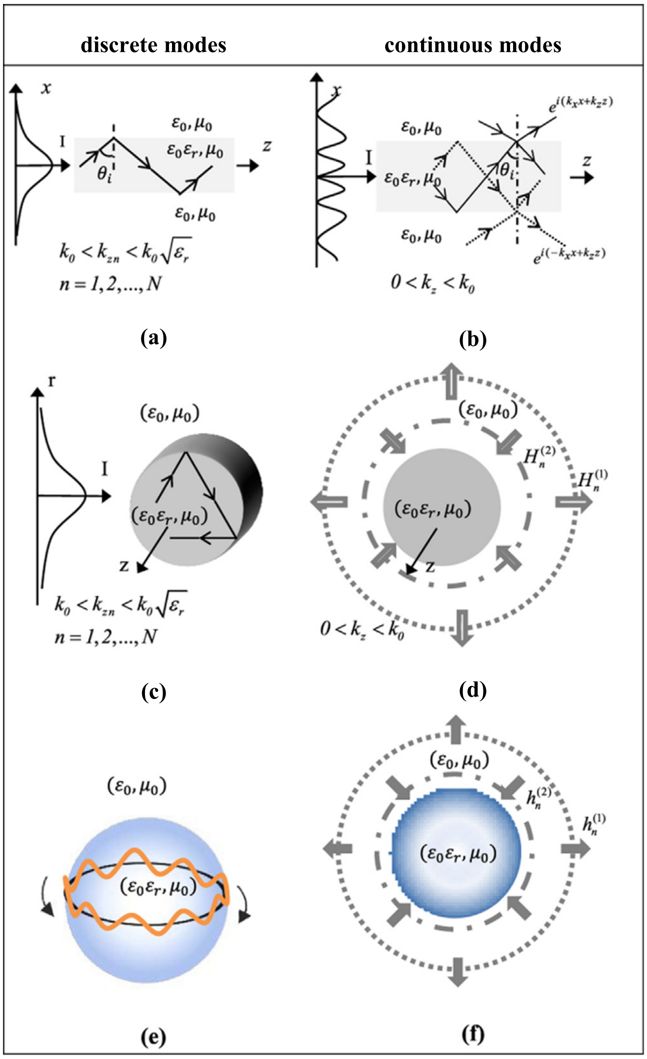
Table 1
Discrete and continuous modes in dielectric slab (a–b) and cylindrical waveguide (c–d). Discrete and continuous spectrum of resonance modes of a spherical dielectric resonator (e–f).



loss.

The rest of the paper is organized as follows. In Section 2, the characteristic equation of the dielectric resonator is presented. Section 3 concerns with numerical results for the resonator modes and their quality factors along with some discussions. Finally, some concluding remarks are the subject of Section 4. In Appendix A and B, details of deriving an analytical formulation for the radiating and evanescent waves are presented.

2. Analytical formulation

Consider a dielectric sphere of radius $r = a$ made of a homogeneous, isotropic, and reciprocal material with a complex relative permittivity of $\epsilon_1 = \epsilon_r(1 + i\tan(\delta))$ and a relative permeability of $\mu_1 = 1$ being surrounded by vacuum ($\varepsilon_2 = 1$ and $\mu_2 = 1$). The center of this sphere coincides with the origin of a spherical coordinate system (r, θ, φ). Furthermore, time-harmonic variation of the form $\exp(-i\,\omega\,t)$ is assumed. The method of separation of variables leads to solutions of the Helmholtz equation in the form of

$$\psi_{m,n}^{q}(r, \theta, \varphi) = P_n^m(\cos(\theta)) \begin{Bmatrix} \sin(m\,\varphi) \\ \text{or} \\ \cos(m\,\varphi) \end{Bmatrix} k_q\, r\, b_n(k_q\, r) \tag{1}$$

where $P_n^m(\cos(\theta))$ is the associated Legendre functions of the first kind which are complete and orthogonal over $[0, \pi]$, as detailed in [17]. Furthermore, the trigonometric functions $\sin(m\,\varphi)$ or $\cos(m\,\varphi)$ are also complete and orthogonal over $[0, 2\pi]$. So, the functions $\psi_{m,n}^{q}(r, \theta, \varphi)$ form the potential functions required for generating transverse electric TE_r or transverse magnetic TM_r fields inside ($q = 1$) and outside ($q = 2$) the sphere. The parameters n and m are integers representing terrestrial harmonics and k is the wave number which can assume the values

$$k_q = \frac{\omega}{c}\sqrt{\varepsilon_q}, \quad q = 1, 2 \tag{2}$$

Here, c denotes speed of light in free space. So, we assume

$$\begin{cases} b_n(k_1 r) = j_n(k_1 r) & r < a \\ b_n(k_2 r) = C_1 h_n^{(1)}(k_2 r) + C_2 h_n^{(2)}(k_2 r) & r \geq a \end{cases} \tag{3}$$

for the spherical Bessel functions appearing in Eq. (1). Note that the potential function outside the sphere has been expressed by Eq. (3) as a superposition of outward and inward traveling waves. By substituting Eq. (3) in Eq. (1) and applying the boundary conditions at $r = a$, the characteristic equation for the resonator modes is

$$p\frac{r\,j_n(k_1\,r)}{\frac{d}{dr}\left[k_1\,r\,j_n(k_1\,r)\right]} = \left.\frac{r\,b_n(k_2\,r)}{\frac{d}{dr}\left[k_2\,r\,b_n(k_2\,r)\right]}\right|_{\omega_{mnl}} \tag{4}$$

where $p = \frac{\varepsilon_1}{\varepsilon_2}$ and $p = 1$ for TM_r and TE_r modes, respectively. Suppose that the angular frequency ω_{mnl} is the l-th root of Eq. (4). We define n and l as the resonance 'order' and the resonance 'rank', respectively. Here, two assumptions can be made for the description of potential function outside the dielectric sphere:

A. A single Hankel function by choosing either $C_1 = 0$ or $C_2 = 0$.
B. Evanescent waves by proper selection of $C_1,\ C_2 \neq 0$.

In the following, it is shown that the first assumption cannot be valid, and the resonator modes are to be searched for in the second group. It is worth mentioning that for a given resonance frequency ω_{mnl}, the coefficients C_1 and C_2 are obtained after applying the two continuity boundary conditions for E_θ and H_θ. Here, the continuity boundary conditions of E_φ and H_φ are also satisfied.

2.1. Assumption of a single Hankel function

In this subsection, we assume $C_1 = 0$ or $C_2 = 0$. Therefore, $b_n(k_2 r)$ is reduced to either an inward or outward traveling spherical Hankel function. Similar to [7–11], we choose the spherical Hankel function of the first kind

$$b_n(k_2 r) = h_n^{(1)}(k_2\ r), \quad r \geq a \tag{5}$$

which represents an outward traveling wave. Here, we seek *real* resonance frequencies ω_{mnl} for a loss-less dielectric, i.e.,

$$\tan(\delta) = 0 \tag{6}$$

By substituting Eq. (5) in Eq. (4), the left-hand side of this equation remains real-valued while, in general, the right-hand side is complex-valued. In other words, the imaginary part of the right-hand side of Eq. (4) must be zero at the resonance frequencies. In Appendix A, it is shown that this happens only if $k_2 r = 0$, which means that the assumption of an outward traveling