Optics Communications 366 (2016) 17–21

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Invited Paper

Optimal representation and processing of optical signals in quadratic-phase systems

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ARTICLE INFO

Article history:
Received 6 November 2015
Received in revised form
11 December 2015
Accepted 12 December 2015
Available online 24 December 2015

Keywords:
ABCD systems
Quadratic-phase systems
Fourier optics
Sampling
Fractional Fourier transform
Linear canonical transform

ABSTRACT

Optical fields propagating through quadratic-phase systems (QPSs) can be modeled as magnified fractional Fourier transforms (FRTs) of the input field, provided we observe them on suitably defined spherical reference surfaces. Non-redundant representation of the fields with the minimum number of samples becomes possible by appropriate choice of sample points on these surfaces. Longitudinally, these surfaces should not be spaced equally with the distance of propagation, but with respect to the FRT order. The non-uniform sampling grid that emerges mirrors the fundamental structure of propagation through QPSs. By providing a means to effectively handle the sampling of chirp functions, it allows for accurate and efficient computation of optical fields propagating in QPSs.

1. Introduction

Quadratic-phase systems (QPSs), also known as first-order optical systems or ABCD systems, are a very general family of optical systems encompassing arbitrary concatenations of various components such as thin lenses and sections of free space in the Fresnel approximation, as well as quadratic graded-index media [1–4]. Mathematically, QPSs are referred to as linear canonical transforms [5,7,6,8–11]. In this paper, we derive the optimal sampling grid for this general family. This non-redundant grid of sample points mirrors the physical structure of QPSs and enables their accurate and fast simulation. The fractional Fourier transform (FRT) plays a fundamental role in the analysis of QPSs. We also analyze the evolution of spatial information along the longitudinal direction and show that the spherical reference surfaces should be equally spaced with respect to the fractional Fourier transform order. Our results are relevant to work on both sampling and fast computation of light fields propagating through quadratic-phase systems [12–26].

2. Decomposition of propagation in quadratic-phase systems

We will use $\hat{f}(x)$ and $\hat{F}(\sigma_x)$ to represent an optical signal in the space domain and the frequency domain, respectively. Although we work with functions of a single variable for sake of simpler analysis, our results can be generalized to two dimensions. It will be useful to introduce dimensionless variables $u = x/s$ and $\mu = s\sigma_x$ for space and frequency, where s is a scaling parameter with units of length. We now define $\hat{f}(x) \equiv (1/\sqrt{s}) f(u)$ and $\hat{F}(\sigma_x) \equiv \sqrt{s}\, F(\mu)$. The functions $f(u)$ and $F(\mu)$ are the space- and frequency-domain functions that take dimensionless arguments. More information on this dimensional normalization process may be found in [1].

The FRT can be viewed as the "fractional operator power" of the common Fourier transform (FT). One way of defining the ath order FRT of a function $f(u)$, which we denote by $f_a(u)$, is

$$f_a(u) = A_a \int_{-\infty}^{\infty} \exp[i\pi(u^2 \cot(a\pi/2) - 2uu' \csc(a\pi/2) + u'^2 \cot(a\pi/2))] f(u') du' \quad (1)$$

See [1] for subtleties in the definition of fractional operator powers as well as alternative ways of defining the FRT. Here $A_a = \sqrt{1 - i\cot(a\pi/2)}$. When $a=1$, we obtain the common FT, so that the FRT can be seen as a generalization of the common FT. Viewed in the space–frequency plane (phase space), the act of taking the FRT of a signal results in a $\alpha = a\pi/2$ rotation of its Wigner distribution. This can be expressed as a relation between the Wigner distribution of $f(u)$ and the Wigner distribution of $f_a(u)$ as follows [27]:

$$W_{f_a}(u, \mu) = W_f(u \cos\alpha - \mu \sin\alpha, u \sin\alpha + \mu \cos\alpha). \quad (2)$$

Quadratic-phase systems (QPSs) are unitary. The input $\hat{f}(x)$ leads to an output $\hat{g}(x)$ given by

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http://dx.doi.org/10.1016/j.optcom.2015.12.025
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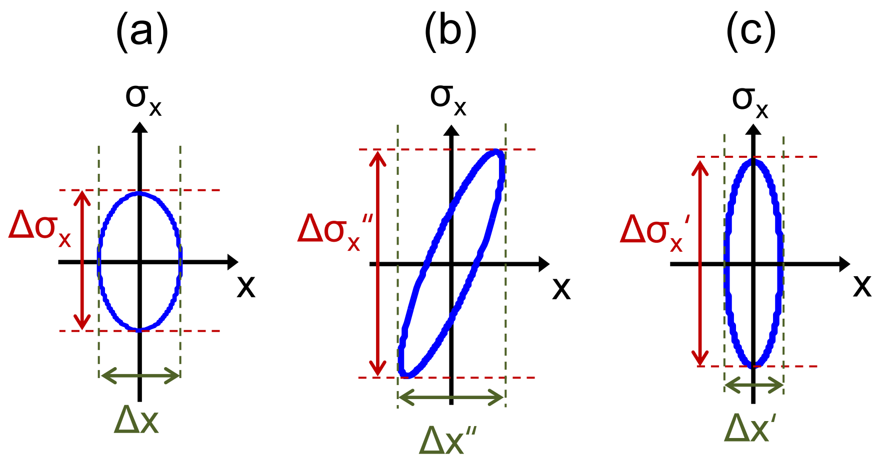

Fig. 1. The space–frequency ellipses show the approximate region of confinement of the (a) input signal, (b) output signal on the planar reference surface, and (c) output signal on the spherical reference surface ($s = \sqrt{\Delta x/\Delta\sigma_x}$, which is the optimal value).

$$\hat{g}(x) = \sqrt{\beta}\, e^{-i\pi/4} \int_{-\infty}^{\infty} e^{i\pi(\alpha x^2 - 2\beta x x' + \gamma x'^2)} \hat{f}(x') dx'. \tag{3}$$

They are often characterized by the *ABCD* parameters which satisfy $AD - BC = 1$ and are defined as

$$A = \frac{\gamma}{\beta}, \quad B = \frac{1}{\beta}, \quad C = -\beta + \frac{\gamma\alpha}{\beta}, \quad D = \frac{\alpha}{\beta}. \tag{4}$$

Consider a quasi-monochromatic optical signal with wavelength λ. Fresnel diffraction and passage through a thin lens are special kinds of QPSs. For Fresnel propagation over a distance d, we have $A = D = 1$, $C=0$, and $B = \lambda d$. For a thin lens with focal length f we have $A = D = 1$, $B=0$, and $C = -1/\lambda f$.

Quadratic-phase systems have many decompositions. A decomposition is the breaking down of the system into consecutive simpler parts. One of the possible decompositions involves three stages. The first is a FRT operation, the second is a magnification operation, and the final stage is a chirp multiplication operation [28–32]:

$$\hat{g}(x) = e^{i2\pi d/\lambda} e^{-ia\pi/4} \sqrt{\frac{1}{sM}} \exp\left(\frac{i\pi x^2}{\lambda R}\right) f_a\left(\frac{x}{sM}\right). \tag{5}$$

Here a, M, and R are defined through

$$\tan\left(\frac{a\pi}{2}\right) = \frac{1}{s^2}\frac{B}{A}, \tag{6}$$

$$M = \sqrt{A^2 + \frac{B^2}{s^4}} = \left|\frac{A}{\cos(a\pi/2)}\right|, \tag{7}$$

$$\frac{1}{\lambda R} = \frac{1}{s^4}\frac{B/A}{A^2 + B^2/s^4} + \frac{C}{A}. \tag{8}$$

The ambiguity in inverting the tangent in Eq. (6) should be resolved by choosing a in $0 < a < 2$ for $B > 0$ and in $2 < a < 4$ for $B < 0$. We will make a number of observations on these equations. First, we note that the unit-magnitude $e^{i2\pi d/\lambda} e^{-ia\pi/4}$ terms are constant and unimportant for our purposes. Furthermore, the chirp multiplication term can be eliminated, if we decide to observe the output not on a planar surface, but on a spherical reference surface with radius as given by the value of R above. In this case, we observe a magnified version of the FRT of the input, with the magnification given by the value of M above. A final note is that, Eq. (5) is valid no matter what we choose s to be. This decomposition will constitute the basis for our derivation of the optimal sampling grid.

3. Transverse sampling spacing

A discussion of sampling often begins with assumptions on the extent of the signals in both space and frequency, the latter often called the bandwidth. An alternative approach is to specify the extent of the signals in the space–frequency plane. Our beginning assumption will be to specify the space–frequency region to which the signal is confined. Here, “confined” means that a sufficiently large percentage of the total energy is contained in that region.

We will take the $z=0$ plane as our input plane. We assume that the space–frequency region of confinement of the input signal at this plane is an ellipse with diameters denoted by Δx and $\Delta\sigma_x$ (Fig. 1(a)). Note that this implies that the space extent of the signal is Δx and that the frequency extent of the signal is $\Delta\sigma_x$. How many samples are needed to represent the signal? The sampling theorem of Nyquist–Shannon requires a sampling interval of $1/\Delta\sigma_x$. Over a spatial extent of Δx this means $N = \Delta x/(1/\Delta\sigma_x) = \Delta x \Delta\sigma_x$ samples. This is known as the space–bandwidth product. The same derivation can be repeated in dimensionless coordinates. Now the ellipse diameters (and thus also the space and frequency extents) are $\Delta x/s$ and $s\Delta\sigma_x$, which lead to the same value of N. The space–bandwidth product is invariant under scalings and does not depend on s.

We now consider a QPS between the $z=0$ and $z=d$ planes characterized by the parameters *ABCD*. For example, in a system made up of lenses separated by sections of free space, these *ABCD* parameters will depend on the focal lengths and locations of the lenses. We will first determine the spatial extent of the output signal $\hat{g}(x)$ observed at $z=d$, given an input signal $\hat{f}(x)$ at $z=0$. It is known that the Wigner distributions of $\hat{f}(x)$ and $\hat{g}(x)$ have the following relationship [1,33–36]:

$$\hat{W}_g(x, \sigma_x) = \hat{W}_f(Dx - B\sigma_x, -Cx + A\sigma_x). \tag{9}$$

Based on our assumption that the initial space–frequency distribution of the signal is well-confined to an elliptical region with diameters Δx and $\Delta\sigma_x$, and using Eq. (9), it is possible to show that the output space–frequency distribution will have a spatial extent

$$\Delta x'' = \sqrt{\Delta x^2 D^2 + \Delta\sigma_x^2 B^2}, \tag{10}$$