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Non-paraxial dispersive shock-waves

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ABSTRACT

We investigate the effect of non-paraxiality in the dynamics of spatial dispersive shock waves in the defocusing nonlinear Schrödinger equation. We find that the lowest order correction in the degree on non-paraxiality enhances the wave-breaking and imposes a limit to the highest achievable spatial spectral content generated by the shocks.

1. Introduction

Dispersive shock waves (DSWs) have been the subject of intense research in the field of nonlinear waves [1,2], with specific applications in Bose–Einstein condensation [3–7] and nonlinear optics [8–11], and are part of the large number of hydrodynamic-like phenomena [12] that are considered important because of their links with quantum fluids [13], turbulence [14,15], disordered and curved systems [16,17], and their application to laser physics [18].

With specific reference to nonlinear optics, all the reported theoretical investigations in the spatial domain are based on the paraxial approximation of the propagation equation of the electromagnetic field (as for example considered in [12,16,19,20]). However shock waves are highly nonlinear processes that induce a substantial amount of spectral broadening. One can hence expect that non-paraxial terms are relevant in the development of the wave-breaking phenomena because waves travelling at large angles with respect to the propagation direction can be generated by the excitation of steep spatial wavefronts.

Non-paraxiality was previously investigated in the formation of solitons [21-27], on the contrary, non-paraxial DSWs have not been considered.

An open issue is the effect of non-paraxiality on the features arising from the wave-breaking phenomena as, in particular, the spectral content and the position in which the shock is formed (shock point). More in general, one can argue about the effect of the non-paraxial terms in the hydrodynamic model that is often considered to theoretical analyses of the shock generation.

In this paper, we investigate theoretically and numerically the effect of non-paraxiality in the DSWs. The approach is based on theory of the characteristic lines that turn our to be formally identical to the trajectories of a massive classical particle with first order relativistic corrections to the Newton law. The analysis allows to quantify the limitations to the spatial spectral broadening induced by the non-paraxial corrections to the nonlinear Schroedinger equation, and the corresponding variation of the shock point.

This paper is organized as follows. In Section 2, we review the leading model and the derivation of the hydrodynamic limit. In Section 3, we analyze the hydrodynamic regime and derive the equations for the characteristic lineas in the one-dimensional case. In Section 4, we report the numerical simulations of the leading model in the two-dimensional (2D) case. Conclusions are drawn in Section 5.

2. Model

At the lowest order of perturbation the non-paraxial correction to the Foch–Leontovich equation for a paraxial beam described by a complex envelope A, normalized such that $|A|^2$ is the optical intensity, is written as [25]

$$i\frac{\partial A}{\partial Z} + \left(\frac{\nabla^2_{X,Y}}{2k} - \frac{\nabla^4_{X,Y}}{8k^3}\right)A + k\frac{n_2}{n_0}|A|^2A = 0, \tag{1}$$

letting λ be the wavelength, $k = 2\pi n_0/\lambda$ the wavenumber, n_0 the

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bulk refractive index, and taking a nonlinear Kerr medium, with refracting index perturbation $n_2|A|^2$. In Eq. (1) we neglect vectorial corrections to the nonlinear term [28,23], as we make reference to highly nonlinear processes, such as thermal effects and electrostrictive nonlinearity, for which vectorial effects are known to be negligible [29,30].

We consider the evolution of a focused Gaussian beam with profile at Z=0, $A=\sqrt{I_0}\exp(-X^2/4w_0^2-Y^2/4w_0^2)$ where w_0 is the beam waist. By introducing the scaled coordinates $(x', y', z')=(X/w_0, Y/w_0, Z/L_d)$, with $L_d=kw_0^2$ the diffraction length, and the normalized variable $\psi=A/\sqrt{I_0}$, Eq. (1) can be conveniently rewritten as follows, when $n_2<0$:

$$i\frac{\partial\psi}{\partial z'}+\left[\frac{(\nabla'_\perp)^2}{2}-\varepsilon\frac{(\nabla'_\perp)^4}{8}\right]\psi-|\psi|^2\psi=0, \tag{2}$$

where $(\nabla'_\perp)^2=(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2})$, $\varepsilon=\frac{1}{kL_d}=\frac{\lambda^2}{4\pi^2w_0^2}$, having chosen $I_0=\frac{n_0}{kL_d\,|n_2|}$.

By writing the normalized field as $\psi(r', z')=\sqrt{\rho(r', z')}\exp[i\phi(r', z')]$, where $r'=\sqrt{x'^2+y'^2}$, we can study Eq. (2) in the framework of the WKB approximation [31-33].

In order to resort to the hydrodynamic approximation we introduce a small scaling factor η such that $\psi(r, z)=\sqrt{\rho(r, z)}\exp[i\phi(r, z)/\eta]$, $z\to z'/\eta$, and $(x, y)\to(x'/\eta, y'/\eta)$; substituting in Eq. (2) we obtain

$$i\eta\psi_z+\frac{\eta^2}{2}\nabla_\perp^2\psi-\varepsilon\frac{\eta^4}{8}\nabla_\perp^4\psi-|\psi|^2\psi=0 \tag{3}$$

where $\nabla_\perp^2=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$.

Analytical treatment of the problem under consideration can be done in hydrodynamical approximation in 1D case as detailed in the following. As we are interested to the experimentally relevant 2D case, we compare in a later section the following theoretical results with 2D numerical simulations of Eq. (2).

At the lowest order in η, the hydrodynamical approximation prescribes a density $\rho=\rho(x)$ independent by the propagation direction z, hence Eq. (3) reduces to the following equation for the phase ϕ:

$$\phi_z+\frac{1}{2}\phi_x^2+\frac{\varepsilon}{8}\phi_x^4=-\rho(x). \tag{4}$$

By defining a velocity field as $v=\phi_x$ and differentiating w.r.t. z, we obtain the following equation:

$$v_z+vv_x+\frac{\varepsilon}{2}v^3v_x=-\partial_x\rho(x). \tag{5}$$

We notice that Eq. (5) is formally similar to the Hopf equation, the solutions of which are known to develop the wave-breaking phenomenon [34]. Non-paraxiality induces the higher order term v^3v_x.

3. Analysis of characteristic lines

The wave-breaking phenomenon supported by Eq. (5) is a consequence of the occurrence, at the shock point, of a singularity in the velocity profile. After the shock point in the hydrodynamical limit, for the nonlinear propagation equation (Eq. (1)) this results into a formation of fast oscillations that regularize the singularity. The mathematical analysis in the hydrodynamical limit is determined in the following by the method of characteristic lines. Such a method (see for example [34]) is a mathematical technique that enables to solve quasi-linear one dimensional partial differential equations (PDEs) by using a system of ordinary differential equations (ODEs). The characteristic lines are the trajectories resulting from ODEs for different initial conditions and give the direction of the energy propagation of the PDE. The point where the characteristic lines cross each other is the shock point, i.e. the point where the solution of the PDE becomes a multi-valued function. In our case this method allows us to express the solution of Eq. (5) in terms of Hamiltonian system of ordinary differential equations:

$$\frac{dv}{dz}=f(x)=-\partial_x\rho(x)=-\frac{\partial H(x, v)}{\partial x}$$
$$\frac{dx}{dz}=v+\frac{\varepsilon}{2}v^3=\frac{\partial H(x, v)}{\partial v}, \tag{6}$$

where $H(x, v)=\frac{v^2}{2}+\frac{\varepsilon}{8}v^4+\rho(x)$ is the conserved Hamiltonian, and $f(x)=-\partial_x\rho(x)$, is a conservative force, with the intensity profile $\rho(x)$ playing the role of the potential. The non-paraxial term, weighted by ε, gives a contribution which resembles the relativistic correction to the motion of a particle. In fact, in special relativity the dynamics of a single particle subject to a conservative force $-\partial_x\rho(x)$ with rest mass, m_0, is given by the Hamiltonian $H_{RL}(x, v)=\gamma c^2+\rho(x)$ with the Lorentz factor $\gamma=1/\sqrt{1-v^2/c^2}$ and c the velocity of light. In the limit $v<<c$, $H_{RL}(x, v)\simeq m_0c^2+\frac{m_0v^2}{2}+\frac{3m_0v^4}{8c^2}+\rho(x)$; in units such that $m_0=1$, this gives the Hamiltonian dynamics (6) with $\varepsilon=3/c^2$.

3.1. Maximal velocity

The analogy with the relativistic dynamics indicates that the effect of non-paraxiality reduces the spatial spectrum resulting from the shock. Indeed the velocity v of the effective particle corresponds to wavector k of an optical ray. By using the conservation of $H(x, v)$ in (6), the case of an input beam with a flat phase front corresponds to an initial distribution particles with zero velocity positioned in a potential $\rho(x)$ given by the intensity profile of the beam. Hence, at z=0 all the particles have a distribution of potential energy $\rho(x)$ that, upon propagation, is converted in kinetic energy. The condition $H(x, v)=v^2/2+\varepsilon v^4/8=\rho(x)$ gives the maximal velocity v_{MAX} of a characteristic line originally placed in x. For the considered Gaussian beam $\rho(x)=\exp(-x^2/2)$, the particles located in proximity of the peak intensity x=0 have the highest velocity and collide upon propagation with those located at the beam edges causing the hydrodynamic shock (see Fig. 1(a)). The conservation of $H(x, v)$ shows that v_{MAX} is reduced when increasing ε:

$$v_{MAX}(\varepsilon)=v_{MAX}(0)\left(1-\frac{\varepsilon}{4}\right)+O(\varepsilon^2), \tag{7}$$

with $v_{MAX}(0)=\sqrt{2}$ in the Gaussian case $\rho(x)=\exp(-x^2/2)$. Eq. (7) predicts that after the shock, non-paraxial effects limit the maximal achievable velocity. The distribution of velocity is directly measurable by the far-field in optical measurements [19].

3.2. Shock point

As shown in Fig. 1(a) the shock is signaled by the caustic resulting from the envelope of characteristic lines at the boundary of the beam. The lines in these regions are parabolic, and starting from a point x_0 with v=0, they can be analytically approximated by solving Eq. (6) with $f(x)=f(x_0)$ approximately constant, which gives an estimate of the shock point z_S. Considering two infinitesimally near characteristic lines starting at $x_0+dx/2$ and $x_0-dx/2$ indicated respectively as $x_+(z)$ and $x_-(z)$, the shock point can be found by the condition $x_+(z)=x_-(z)$. Eqs. (6) can be solved