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Constitutive relations in optics in terms of geometric algebra

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ABSTRACT

To analyze the electromagnetic wave propagation in a medium the Maxwell equations should be supplemented by constitutive relations. At present the classification of linear constitutive relations is well established in tensorial-matrix and exterior *p*-form calculus. Here the constitutive relations are found in the context of Clifford geometric algebra. For this purpose $Cl_{1,3}$ algebra that conforms with relativistic 4D Minkowskian spacetime is used. It is shown that the classification of linear optical phenomena with the help of constitutive relations in this case comes from the structure of $Cl_{1,3}$ algebra itself. Concrete expressions for constitutive relations which follow from this algebra are presented. They can be applied in calculating the propagation properties of electromagnetic waves in any anisotropic, linear and non-dissipative medium.

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1. Introduction

The constitutive relations are needed to connect a pair of vacuum fields (E, B) with a pair of excitation fields (D, H) in a medium in order to close the system of Maxwell equations. The simplest constitutive relations are D=εE and H=μ^{-1}B, where ϵ and μ are the permittivity and permeability. Traditionally the constitutive relations are described at a fixed frequency as a linear transformation between the mentioned pairs of fields [1–3,8,4]. To include memory of the medium or transient properties one should resort to integral formulation with a time-dependent response function in the integrand of the constitutive relation [5,6]. This greatly complicates the problem.

The restriction of the problem to frequency domain and, in addition, to dissipationless linear and unbounded media allows to formulate constitutive relations for a relatively wide class of media, including the pre-metric electrodynamics [3,7]. In tensorial form such formulations are summarized in [2] and exterior *p*-form calculus in [3], where it is shown that in the most general case the constitutive relation is characterized by 36 scalar coefficients. Dyadic approach to the problem which is frequently met in electrical engineering is summarized in [4,9]. The dyads simplify tensorial notation and subsequent calculations. However they are not the objects of a linear (vector) space. Here the situation may be compared with introduction of classical magnetic field vector B, which does not belong to basis vectors of linear space of algebra. The price one pays is that now two kinds of vectors, polar and axial, having different transformation properties appear. In geometric algebra (GA) the objects of linear space are vectors, bivectors (oriented planes), trivectors (oriented volumes) and in general the multivectors. In GA all linear transformations are limited only between multivectors. Of course, one can introduce a linear transformations for B, or more generally between (multi) dyads at the cost of (unnecessary) complication of mathematics. However, now new rules appear, for example, one is not allowed to sum up linear space forming vectors with the axial vectors, otherwise one will get a monster with strange properties under reflection. New difficulties may appear when dyads are applied in differential and integral calculus. Here the grad, div and curl operators expressed through vectorial nabla operator do not possess the inverses. Whereas in GA, the nabla operator carries the properties of metric space. As a consequence, the differential GA operator has inverse [10]. Thus, useful solutions may be lost if the GA nabla is artificially divided into div and curl.

The Clifford or geometric algebra, where the metric of the spacetime is predetermined, offers a different and more efficient approach to the solved problem. The main advantage of GA is a coordinate-free attitude which is reminiscent of classical vectorial calculus but extended to multidimensional linear spaces with a given space metric. In GA the objects are multivectors of different grades that have geometric interpretation and can be simply manipulated including their transformations between same grade as well as different grade subspaces. The multivectors of GA satisfy a number of outermorphisms (or involutions) which automatically ensure symmetry properties of the spacetime, namely, *P* and *T* transformations of the space and time, and their combination *PT* [11]. Of all 64 irreducible Clifford algebras represented by 8-periodicity table [12], in optics and electrodynamics the two are the

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most important, namely, $Cl_{3,0}$ algebra which describes Euclidean 3D space and $Cl_{1,3}$ algebra which describes Minkowskian relativistic 4D spacetime. Respectively, the optics related with these two algebras may be called classical or Galilean, and relativistic or Minkowskian. There is a number of books of different degrees of difficulty that explain electromagnetic wave propagation in terms of GA [13–16]. An accessible introduction to GA and electrodynamics for readers who are unfamiliar with this subject can be found in T. G. Vold's two articles [17].

The first attempts to construct the constitutive relations in terms of GA multivectors can be found in [6,18–20]. In [6] the general restrictions in the time domain are analyzed. In [19] the constitutive relations for isotropic material are formulated in a covariant manner. In [18] nonorthogonal frame is used to describe the permittivity ellipsoid. In [20] an additive form of constitutive relations is considered where the influence of material is described by polarization of medium. In this paper we use the multiplicative form for constitutive relations that connect the primary (E, B) and the secondary (D, H) electromagnetic fields by a linear transformation of GA.

In this paper, both the Dirac gamma and Pauli sigma notations [15] will be used to designate the multivectors (for definitions see Appendix A). This notation incorporates a link between the Euclidean and Minkowskian spaces, namely, that even subalgebra of $Cl_{1,3}$ is isomorphic to Euclidean space algebra $Cl_{3,0}$ and, therefore, a projection of the relativistic multivector onto even subalgebra yields measured in experiment quantities.

In the next Section 2, the electromagnetic (EM) field properties in $Cl_{1,3}$ algebra are reminded. In Section 3 properties of linear transformations used in construction of constitutive relations are briefly discussed. General relativistic constitutive relations in GA are presented in Section 4. A summary of some of definitions used in GA and properties of multivectors in $Cl_{1,3}$ are summarized in Appendix Appendix A.

2. The EM fields and excitations in $Cl_{1,3}$

The geometric algebra $Cl_{1,3}$ is characterized by $(+,-,-,-)$ metric. This algebra describes relativistic spacetime mapped by four orthogonal basis vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ which are isomorphic to Dirac matrices. The squares of γ_i's give the metric of the spacetime, i.e. $\gamma_0^2 = 1$ and $\gamma_i^2 = -1$ for $i = 1, 2, 3$. Since in the relativity the primary EM field F as well as the secondary field G (the latter according to [3] will be called the excitation) are $Cl_{1,3}$ algebra bivectors, the constitutive relations make a set of linear GA transformations which connect different combinations of elementary bivectors.

The primary electromagnetic field F (Faraday bivector) can be decomposed into six elementary bivectors that represent six oriented planes in the Minkowskian 4D space,

$$\mathrm{F}=\mathrm{F}^{(-)} + \mathrm{F}^{(+)}, \tag{1}$$

$$\mathrm{F}^{(-)} \equiv \mathrm{E}= E_1\boldsymbol{\sigma}_1 + E_2\boldsymbol{\sigma}_2 + E_3\boldsymbol{\sigma}_3, \tag{2}$$

$$\mathrm{F}^{(+)} \equiv \mathrm{B}= B_1 I\boldsymbol{\sigma}_1 + B_2 I\boldsymbol{\sigma}_2 + B_3 I\boldsymbol{\sigma}_3. \tag{3}$$

Plus and minus signs indicate even and odd parts with respect to spatial inversion indicated by overline, $\overline{\mathrm{F}}^{(+)} = \mathrm{F}^{(+)}$ and $\overline{\mathrm{F}}^{(-)} = -\mathrm{F}^{(-)}$ (Appendix A). The bivectors $\boldsymbol{\sigma}_i \equiv \gamma_i\gamma_0$, where γ_0 is the time coordinate and γ_i are the space coordinates, are time-like (odd with respect to spatial inversion) the squares of which give $\boldsymbol{\sigma}_i^2 = \gamma_0\gamma_i\gamma_0\gamma_i = -\gamma_i\gamma_0\gamma_0\gamma_i = -\gamma_i\gamma_i = +1$. The bivectors $I\boldsymbol{\sigma}_1 = \gamma_3\gamma_2 = -\gamma_2\gamma_3$, $I\boldsymbol{\sigma}_2 = \gamma_1\gamma_3 = -\gamma_3\gamma_1$ and $I\boldsymbol{\sigma}_3 = \gamma_2\gamma_1 = -\gamma_1\gamma_2$ are space-like (even with respect to spatial inversion) the square of which simplify to -1. From all this follows that the squares of electric and magnetic field components satisfy $\mathrm{E}^2 > 0$ and $\mathrm{B}^2 < 0$.

The other pair of fields, called the excitations D and H, depends on material properties and is written in a similar manner,

$$\mathrm{G}=\mathrm{G}^{(+)} + \mathrm{G}^{(-)}, \tag{4}$$

$$\mathrm{G}^{(+)} \equiv \mathrm{D}= D_1\boldsymbol{\sigma}_1 + D_2\boldsymbol{\sigma}_2 + D_3\boldsymbol{\sigma}_3, \tag{5}$$

$$\mathrm{G}^{(-)} \equiv \mathrm{H}= H_1 I\boldsymbol{\sigma}_1 + H_2 I\boldsymbol{\sigma}_2 + H_3 I\boldsymbol{\sigma}_3. \tag{6}$$

Similarly, the excitations satisfy $\mathrm{D}^2 > 0$ and $\mathrm{H}^2 < 0$.

It is assumed that the medium is lossless, linear, and unbounded, with instantaneous response to external fields. Then the constitutive relation between the Faraday field F and excitation field G is determined by operator $\hat{\chi}$,

$$\mathrm{G}= \mathrm{D}+ \mathrm{H}= \hat{\chi}(\mathrm{E}+ \mathrm{B}) = \hat{\chi}\mathrm{F}, \tag{7}$$

which is a linear bivector-valued function of the bivector argument. Also, we shall assume that the constitutive relation between F and G is local. We adopt that the vacuum electromagnetic constants are normalized, $\varepsilon_0 = \mu_0 = 1$, so that Eq. (7) is dimensionless. The dimensions of fields in (7) and subsequent equations in SI system can be recovered referring to the following dimensional relation between the fields

$$\begin{aligned}[\mathrm{D}] &= [\varepsilon_0][\mathrm{E}] + [Y_0][\mathrm{B}],\\ [\mathrm{H}] &= [Y_0][\mathrm{E}] + [\mu_0^{-1}][\mathrm{B}],\end{aligned} \tag{8}$$

where $Y_0 = (\varepsilon_0/\mu_0)^{1/2}$ is the admittance of the vacuum. The terms with ε_0 and μ_0 correspond to usual relations between the fields in the medium, while the terms with Y_0 are called magnetoelectric components.

The bivector transformation (7) closes the pair of Maxwell equations, which in $Cl_{1,3}$ algebra read [15]

$$\begin{aligned}\nabla \wedge \mathrm{F}&= 0,\\ \nabla\cdot \mathrm{G}&= J,\end{aligned} \tag{9}$$

where J is the 4-current. In this article it is equated to zero. ∇ is vectorial differential operator $\nabla = \gamma_0\partial_t - \gamma_1\partial_x - \gamma_2\partial_y - \gamma_3\partial_z$, in which the metric of the Minkowskian space is embodied. The Maxwell Eqs. (9) are written without reference to any particular frame. In case of the vacuum they can be reduced to a single equation [15]. However, for a general constitutive relation one has to work with both coupled equations.

The constitutive relation (7) in GA is equivalent to the tensorial relation obtained by E.J. Post [2]

$$G^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}, \tag{10}$$

where χ^{ijkl} is a constitutive tensor density of rank 4 which satisfies the following symmetry relations $\chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk}$. As shown in [2], due to spacetime symmetry, out of 256 components of χ^{ijkl} there remain only 36 independent ones which for convenience can be cast into form of 6×6 matrix. In GA formulation no additional spacetime symmetry considerations are needed. As shown in the paper [11] the identity, inversion, reversion and Clifford conjugation operations in GA are isomorphic to group of four, which consists of identity operation, space P and time T reversals, and the combination PT. Thus in geometric algebra the symmetry operations $\{1, P, T, PT\}$ are automatically satisfied. These are called the fundamental involutions (automorphisms) of GA. Note, that the operations are isomorphic, but not identical as names imply. Thus, in GA no additional requirements are needed to find the constitutive relations between fields and excitations.

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