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Plane-wave expansion of elliptic cylindrical functions

Carlo Santini *, Fabrizio Frezza, Nicola Tedeschi

Department of Information Engineering, Electronics and Telecommunications, "La Sapienza" University of Rome, Via Eudossiana 18, 00184 Rome, Italy

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A B S T R A C T

Elliptic Cylindrical Waves (ECW), defined as the product of an angular Mathieu function by its corresponding radial Mathieu function, occur in the solution of scattering problems involving two-dimensional structures with elliptic cross sections. In this paper, we explicitly derive the expansion of ECW, along a plane surface, in terms of homogeneous and evanescent plane waves, showing the accuracy of the numerical implementation of the formulas and discussing possible applications of the result.

1. Introduction

The plane-wave representation of electromagnetic field, in its connection to Fourier analysis, is a fundamental tool in dealing with several aspects of the electromagnetic theory [1,2]. By expressing complex electromagnetic fields in terms of superpositions of very simple solutions of Maxwell's equations, it is capable of delivering a great simplification in the analytical treatment of several complex radiation, propagation and diffraction problems. In particular, the aforementioned technique may be used to express the electromagnetic field radiated by localized sources or scattered by localized obstacles with simple shapes, expressing the typical solution of Helmholtz equation in orthogonal curvilinear coordinates in terms of natural solutions of Maxwell's equations in Cartesian coordinates [2,3]. This approach has proved to be extremely fruitful in dealing with the reflection of complex electromagnetic fields by plane surfaces, e.g., when the fields are expanded in terms of cylindrical functions in circular coordinates [4–8]. Since the reflection and transmission properties of surfaces are known, or at least easily expressible, just for incident plane waves, solutions of diffraction problems in the presence of a generally reflecting plane surface require an integral expansion of the diffracted field along the plane surface in terms of homogeneous and evanescent plane waves, for which the reflection behavior may be characterized by means of the Fresnel coefficients [9–12]. The solution of two-dimensional scattering problems in elliptic coordinates is pursued by expanding the diffracted field by means of Elliptic Cylindrical Waves (ECW), defined as the product of an angular Mathieu function by its corresponding radial Mathieu function. Integral plane-wave representations of ECW as a contour integral in the complex plane may be found in many fundamental works [2,13,14] but, to the best of our knowledge, none of the available forms is suitable for the straightforward application of the aforementioned analytical procedure.

In this paper we show the explicit derivation of the plane-wave spectrum of ECW, whose final analytical form is directly applicable to the study of the reflection of ECW by a planar discontinuity between propagation media. Such result is significant in many fields of applied optics, since it constitutes a basilar step in the construction of full-wave solutions of scattering problems regarding cylindrical diffracting structures with elliptic cross sections.

The paper is organized as follows: in Section 2 we resume some fundamental concepts about ECW, we define the notations used in this paper and we show explicit analytical derivation of the plane-wave expansion. In Section 3 we present numerical results, pointing out the accuracy and reliability of the proposed plane-wave spectrum representation. In Section 4 we discuss relevant applications of the proposed expansion. Finally, conclusions are given in Section 5, where further developments are outlined too.

* Corresponding author. Fax: +39 06 44585918.
E-mail addresses: carlo.santini@diet.uniroma1.it, carlo1.santini@gmail.com (C. Santini).
URL: http://151.100.120.244/personale/frezza/ (F. Frezza).

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2. Plane wave expansion of ECW

2.1. Elliptic Cylindrical Waves

With reference to the notation used in [3,15], we will denote with symbols $S_{p_n}(v, q)$ the angular Mathieu functions (AMF), and with symbols $J_{p_n}(u, q)$, $N_{p_n}(u, q)$ the radial Mathieu functions (RMF) of the first and second kind, respectively: (u, v) being the elliptic cylindrical coordinates, q being the elliptic parameter, the index $p = \{e, o\}$ and index $n \in \mathbb{N}$ denoting functions of *even* or *odd* type p and integer order n, respectively. With symbols $H_{p_n}^{(m)}(u, q)$, $m = \{1, 2\}$, we will denote radial Mathieu functions of the third kind, analogous to the Hankel functions in circular coordinates, defined as

$$H_{p_n}^{(1)}(u, q) = J_{p_n}(u, q) + iN_{p_n}(u, q), \tag{1}$$

$$H_{p_n}^{(2)}(u, q) = J_{p_n}(u, q) - iN_{p_n}(u, q). \tag{2}$$

By means of such notation, the basic solutions of the Helmholtz equation in elliptic coordinates are of the form

$$H_{p_n}^{(m)}(u, q) \cdot S_{p_n}(v, q);$$

the elliptic parameter q is connected to the wavenumber k in the Helmholtz equation since $q = k^2\rho^2/4$, $k = 2\pi/\lambda$, λ being the wavelength, $\rho = d/2$, d being the interfocal distance of the reference ellipses. For the sake of simplicity and readability in the rest of this paper, we will refer to these basic solutions as Elliptic Cylindrical Waves (ECW), denoted by symbols $\mathrm{ECW}_{p_n}^{(m)}(u, v, q)$, to focus on the analogies with the results of plane-wave expansion of circular cylindrical waves in [4]. We point out that the two different forms of Mathieu functions of the third kind in (1) and (2), also called Mathieu–Hankel functions of the first and second kind, respectively, give rise to corresponding ECW representing outgoing (first kind, $\mathrm{ECW}_{p_n}^{(1)}$) and ingoing (second kind, $\mathrm{ECW}_{p_n}^{(2)}$) fields when a time factor $\exp(-i\omega t)$ is assumed. In this paper, for relevant applications to diffraction theory, we will focus on $\mathrm{ECW}_{p_n}^{(1)}$ functions corresponding to Mathieu–Hankel functions of the first kind:

$$\mathrm{ECW}_{p_n}^{(1)}(u, v, q) = H_{p_n}^{(1)}(u, q) \cdot S_{p_n}(v, q); \tag{3}$$

for this reason, in the following sections, the superscript "(1)" will be dropped.

2.2. Integral representation of ECW

The geometric layout of the problem is shown in Fig. 1, where the axes and coordinates of the Cartesian reference frame are visualized together with the corresponding elliptic coordinates. We will refer to dimensionless Cartesian coordinates $\xi = kx$ and $\eta = ky$. Our aim is to express the ECW field distribution (3), across a plane $\eta = \eta_0 > 0$, as a superposition of plane waves, following an analytical approach similar to the one used in [4]. We start from the integral representation in [14], reported as "Integral Representation with Elementary Kernel" in [16,17] at Section 28.28.7

$$\frac{1}{\pi}\int_{\mathcal{L}} \exp(2ihw)\mathrm{me}_\nu(t, h^2)\, dt = \exp\left(i\nu\frac{\pi}{2}\right)\mathrm{me}_\nu(\alpha, h^2) M_\nu^{(3)}(z, h), \tag{4}$$

where

- variable w is defined as $w = \cosh z \cos t \cos\alpha + \sinh z \sin t \sin\alpha$,
- w, z, α, t represent complex variables,
- $h^2 = q$,
- ν represents a complex index,
- the complex variable t must follow an integration path $\mathcal{L}$

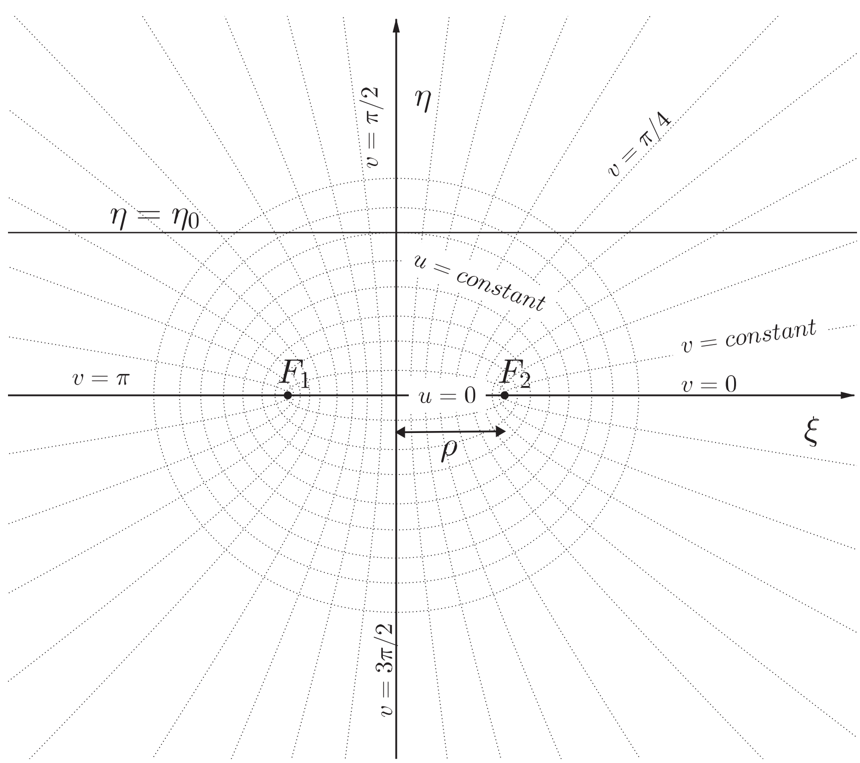

Fig. 1. Geometry of the problem and reference frame; ξ and η are Cartesian dimensionless coordinates, defined as $\xi = kx$ and $\eta = ky$, where $k = 2\pi/\lambda$, λ being the wavelength; (u, v) are elliptic coordinates; F_1 and F_2 are the foci of the elliptic reference frame, $\rho = \overline{F_1F_2}/2$.