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Ultracold two-level atom in a quadratic potential

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ABSTRACT

We use a right unitary decomposition to study an ultracold two-level atom interacting with a quantum field. We show that such a right unitary approach simplifies the numerical evolution for arbitrary position-dependent atom–field couplings. In particular, we provide a closed form, analytic time evolution operator for atom–field couplings with quadratic dependence on the position of the atom; e.g. a two-level atom near an extremum of a cavity field mode amplitude. Our approach allows us to show that the center of mass wave function may be squeezed by choosing a proper atom–field initial state.

1. Introduction

The Jaynes–Cummings (JC) model describing the interaction of a two-level atom with a quantized field mode [1] is a solvable working model of the micromaser [2]. In this model, the center of mass velocity of the two-level atom is slow enough to allow controlled atom by atom interaction with the field but fast enough to be described by classical physics; e.g. thermal Rydberg atoms passing through a superconducting cavity showing Rabi oscillations [3]. In the limit case of a two-level atom so slow that its center of mass motion needs to be quantized, the system is described by the following Hamiltonian [4]:

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \omega\hat{a}^\dagger\hat{a} + \frac{\omega_q}{2}\hat{\sigma}_z + g(\hat{z})\left(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right), \tag{1}$$

where the quantized motion of the two-level atom with unitary mass has been taken in the $\hat{z}$-direction with associated canonical momentum $\hat{p}$, the quantum field is described by the annihilation (creation) operators $\hat{a}$ ($\hat{a}^\dagger$) and the frequency ω, and the inner state of the two-level atom by the Pauli matrices $\hat{\sigma}_j$ with $j = z, +, -$ and the transition frequency ω_q. Two regimes of interest can be identified for this model, depending on the ratio between the atomic kinetic energy and the field–atom interaction energy [5]: the intermediate regime, where the mean atomic kinetic energy is of the order of the mean field–atom interaction energy, and the mazer regime, where the kinetic energy is smaller. Amplification via z-motion induced emission of radiation occurs in the latter and gives origin to the mazer name [5–8]. This model is of interest as cavity quantum electrodynamics (cavity-QED) experiments in these two regimes appear feasible with microwave and optical quantum fields [7,9,10]. Also, it is feasible to control or switch off spin interactions of ultracold atoms trapped in optical lattices [11], as well as to address individual sites of such lattices [12,13] at the moment and, in the near future, it may be possible to couple an individual site to a quantum field as cavity-QED has been demonstrated with Bose–Einstein condensates [14,15].

In the theoretical side of the problem, analytic solutions are known for electromagnetic modes described by sinusoidal and mesa functions [5] and sech^2 function [7]. Also, an adiabatic approximation has been proposed by sinusoidal and Gaussian modes [16]. Here, we introduce a right unitary approach to the problem and provide an analytic solution for a quadratic mode. A quadratic mode may be related to an ultracold two-level atom approaching the maximum of a cavity field in an oblique path or trapped in a sinusoidal optical lattice. In the following section, we introduce the right unitary decomposition of the model Hamiltonian for a general quantum field and construct its time evolution operator. Then, we study the resonant case for quadratic couplings and provide a closed form analytic time evolution operator for the system. At this point, we show that an adequate initial state provides us with a squeezed wave function for the center of mass motion of the atom. Finally, we study the interaction of an ultracold excited atom with number and coherent states of the quantum field.

2. Right unitary decomposition

By moving into the frame defined by the excitation number,

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$\hat{a}^\dagger\hat{a} + \hat{\sigma}_z/2$, rotating at frequency ω, we obtain an interaction picture Hamiltonian,

$$\hat{H}_I = \frac{1}{2}\hat{p}^2 + \frac{\delta}{2}\hat{\sigma}_z + g(\hat{z})\left(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right), \tag{2}$$

where the parameter $\delta = \omega_q - \omega$ is the detuning between the two-level atom and field frequencies. We can follow a right unitary approach [17,18] to decompose this Hamiltonian into the following product:

$$\hat{H}_I = \hat{T}\hat{R}_y\hat{H}_z\hat{R}_y^\dagger\hat{T}^\dagger, \tag{3}$$

where we used a rotation of $\pi/4$ radians around the $\hat{\sigma}_y$ operator,

$$\hat{R}_y = e^{i(\pi/4)\hat{\sigma}_y}, \tag{4}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{5}$$

and the transformation,

$$\hat{T} = \begin{pmatrix} \hat{V} & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{T}^\dagger = \begin{pmatrix} \hat{V}^\dagger & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

The latter is right unitary, $\hat{T}\hat{T}^\dagger = \mathbb{1}$ and $\hat{T}^\dagger\hat{T} \neq \mathbb{1}$, due to the properties of the London exponential of the phase [19,20], also known as Susskind–Glogower [21] operators,

$$\hat{V} = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}, \quad \hat{V}^\dagger = \hat{a}^\dagger\frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}, \tag{7}$$

that yield, in the Fock or number state basis,

$$\hat{V}\hat{V}^\dagger = \mathbb{1}, \tag{8}$$

$$\hat{V}^\dagger\hat{V} = \mathbb{1} - |0\rangle\langle 0|. \tag{9}$$

The new auxiliary Hamiltonian is given by

$$\hat{H}_z = \frac{1}{2}\hat{p}^2 + g(\hat{z})\sqrt{\hat{a}^\dagger\hat{a}}\,\hat{\sigma}_z - \frac{\delta}{2}\hat{\sigma}_x. \tag{10}$$

Typically, a right unitary transformation may act as unitary in just a sector of the corresponding Hilbert space, this is a well known problem in phase operators [22,23]. Here, in order to calculate the evolution operator, $\hat{U}_I(t) = e^{-i\hat{H}_I t}$, it is straightforward to compute each and every term of the corresponding power series to obtain $(\hat{T}\hat{R}_y\hat{H}_z\hat{R}_y^\dagger\hat{T}^\dagger)^j = \hat{T}\hat{R}_y\hat{H}_z^j\hat{R}_y^\dagger\hat{T}^\dagger$ [18]. Thus, the evolution operator of the system is given by the following expression:

$$\hat{U}_I(t) = e^{-i\hat{H}_I t} \tag{11}$$

$$= \hat{T}\hat{R}_y e^{-i\hat{H}_z t}\hat{R}_y^\dagger\hat{T}^\dagger. \tag{12}$$

In other words, the right unitary operators for this Hamiltonian behave like unitary operators in this particular ordering.

In summary, our right-unitary decomposition allows us to construct the time evolution for any given coupling potential for which Eq. (10) is solvable but this does not mean that it is straightforward to interpret the results. In the literature, mazer dynamics for sinusoidal and mesa function [5] and sech^2 [7] are well known. In the following, we will use our approach to solve the quadratic potential mazer and show that it is straightforward to cast the center of mass motion states as squeezed states in this particular case. Furthermore, it seems that a specific operator approach has to be constructed for each and every potential of the form z^j. Thus, the construction of an analytic closed form evolution operator for any given coupling function, $g(\hat{z})$, escapes our efforts at the moment.

3. Time evolution for a quadratic coupling

Here, we will solve the problem for quadratic couplings

$$g_\pm(\hat{z}) = g_0 \pm \frac{|\lambda|}{2}\hat{z}^2. \tag{13}$$

In this case, we can write the auxiliary Hamiltonian in the following form:

$$\hat{H}_{z,\pm} = \begin{pmatrix} \frac{1}{2}\hat{p}^2 + \sqrt{\hat{a}^\dagger\hat{a}}\left(g_0 \pm \frac{\lambda}{2}\hat{z}^2\right) & \delta \\ \delta & \frac{1}{2}\hat{p}^2 - \frac{\lambda}{2}\sqrt{\hat{a}^\dagger\hat{a}}\left(g_0 \pm \frac{\lambda}{2}\hat{z}^2\right) \end{pmatrix}, \tag{14}$$

$$\hat{H}_{z,\pm} = \hat{S}\left[\frac{1}{2}\ln\,\omega(\hat{n})\right]\hat{H}_{0,\pm}\hat{S}^\dagger\left[\frac{1}{2}\ln\,\omega(\hat{n})\right], \tag{15}$$

where the new auxiliary Hamiltonian

$$\hat{H}_{0,\pm} = \begin{pmatrix} \hat{H}_\pm + g_0\sqrt{\hat{n}} & \delta \\ \delta & \hat{H}_\mp - g_0\sqrt{\hat{n}} \end{pmatrix}, \tag{16}$$

contains the standard, $\hat{H}_+ = (\hat{p}^2 + \lambda\hat{z}^2)/2$, and inverted, $\hat{H}_- = (\hat{p}^2 - \lambda\hat{z}^2)/2$, harmonic oscillators, which are equivalent to free propagation and degenerate parametric down-conversion, in that order, or equivalently,

$$\hat{H}_+ = \omega(\hat{n})\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right), \tag{17}$$

$$\hat{H}_- = -\frac{\omega(\hat{n})}{2}\left(\hat{b}^{\dagger 2} + \hat{b}^2\right). \tag{18}$$

Here we defined a frequency in terms of the number operator of the field, $\hat{n} = \hat{a}^\dagger\hat{a}$,

$$\omega\left(\hat{n}\right) = \sqrt{|\lambda|\sqrt{\hat{n}}}, \tag{19}$$

also, we used a boson representation for the atomic center of mass motion

$$\hat{b} = \frac{1}{\sqrt{2}}\left(\hat{z} + i\hat{p}\right), \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{z} - i\hat{p}\right), \tag{20}$$

and the action of the squeeze operators,

$$\hat{S}(\hat{\xi}) = e^{-(1/2)\left(\hat{\xi}\hat{b}^{\dagger 2} - \hat{\xi}^\dagger\hat{b}^2\right)}, \tag{21}$$

where the operator $\hat{\xi}$ acts over the cavity field mode, over the position and momentum operators yield

$$\hat{S}(\hat{\xi})\hat{z}\hat{S}^\dagger(\hat{\xi}) = \hat{z}e^{\hat{\xi}}, \quad \hat{S}(\hat{\xi})\hat{p}\hat{S}^\dagger(\hat{\xi}) = \hat{p}e^{-\hat{\xi}}. \tag{22}$$

Note that each and every Fock state of the field, $|k\rangle_f$, defines a bipartite center of mass-field mode, $\left\{|j\rangle_{CM}|k\rangle_f\right\}$ with $j = 0, 1, 2, \ldots$, and auxiliary frequency $\omega(k) = \sqrt{|\lambda|\sqrt{k}}$.

The time evolution operator of such a model is given by

$$\hat{U}_{I,\pm}(t) = \hat{T}\hat{R}_y\hat{S}e^{-i\hat{H}_{0,\pm}t}\hat{S}^\dagger\hat{R}_y^\dagger\hat{T}^\dagger. \tag{23}$$