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Uncertainty in length conversion due to change of sensitivity coefficients of refractive index

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ABSTRACT

A value of length measured in air should be converted to the value in vacuum for the purpose of comparison. The typical values of the sensitivity coefficients for a particular wavelength (e.g. 632.99 nm) under standard environmental conditions are widely available. Because all measurements are not performed under standard environmental conditions and the sensitivity coefficient is affected by atmospheric pressure, temperature, and humidity, one question arises naturally: how does a measurement change with environmental conditions? This study investigates the uncertainty in length conversion due to the change of sensitivity coefficients of refractive index. We also clarify that there is no significant difference in conversion uncertainties via sensitivity coefficients of phase and group refractive indexes.

1. Introduction

In 2009, a femtosecond optical frequency comb (FOFC) replaced the iodine-stabilized He–Ne laser as the national standard of length in Japan. As a He–Ne laser is a single-frequency source, only the wavelength can be used as a practical standard of length. Because an FOFC is a pulsed laser source, not only the wavelength but also the adjacent pulse repetition interval length (APRIL) can be used to define the meter [1]. The length ruler (the wavelength/the APRIL) times their corresponding frequency parameter (the frequency/the pulse repetition frequency) equals the speed of light in vacuum; the meter is defined in this manner.

Length measurement is usually performed in air. The value of the wavelength/APRIL in air is affected by the phase/group refractive index of air. A value of length measured in air should be converted to the value in vacuum to make it comparable. The refractive index is required for this conversion.

In this conversion, the uncertainty in the refractive index is proportional to the uncertainty in the converted length. The uncertainty in the measurement of a refractive index is affected by sensitivity coefficients and the accuracy of the instrument used for measuring environment parameters (temperature, barometric pressure, and humidity) [2]. The sensitivity coefficient is a quantity that represents the change in the refractive index when an environment parameter is changed. The typical values of the sensitivity coefficient of a particular wavelength (e.g. 632.99 nm) under standard environmental conditions are listed in textbooks [2,3].

Two problems arise naturally. First, we do not know detailed information about the sensitivity coefficient. It is not clear how the sensitivity coefficients are affected by environmental parameters. In other words, we do not know the value of uncertainty in length conversion due to a change in sensitivity coefficients. The second problem is whether there are differences in the conversion uncertainties via sensitivity coefficients of two refractive indexes.

In this work, we show how the uncertainty in length conversion is affected by the change in environmental parameters via the sensitivity coefficients of refractive index. We also compare the phase and group refractive index to test their characteristics in length conversion.

This paper is organized as follows. First, the principle of uncertainty in length measurement due to sensitivity coefficients of refractive index is given in Section 2. Numerical calculations are described in Section 3. Finally, the main conclusions are given in Section 4.

2. Principles

A femtosecond optical frequency comb (FOFC) is a type of pulsed laser. In the frequency domain, an FOFC has a million ($>10^6$) frequency components arranged at equal intervals with the pulse repetition frequency f_{rep}. In the time domain, an FOFC

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emits a pulse train in which pulses are arranged at the APRIL, δ_{vac}. More details can be found elsewhere [4].

In vacuum, the following relation holds between the length ruler γ_{vac} (namely, the wavelength λ_{vac} and the APRIL δ_{vac}) and frequency parameter f_P (frequency f of the wavelength and pulse repetition frequency f_{rep} of APRIL).

$$\gamma_{vac} = c_{vac}/f_P. \tag{1}$$

Here, c_{vac} is the speed of light in vacuum. An arbitrary length in vacuum can be represented as follows:

$$L_{vac} = \left(M_\gamma + N_\gamma\right)\gamma_{vac}, \tag{2}$$

where $M$and N are the integral and the fractional parts, respectively. Subscript γ denotes the measurement unit.

Length measurements are usually performed in air. The following relation holds between the length ruler in air γ_{air} (λ_{air} and δ_{air}) and the length ruler in vacuum γ_{vac} (λ_{vac} and δ_{vac}):

$$\gamma_{air} = \gamma_{vac}/n(\gamma_{vac}). \tag{3}$$

Here, $n(\gamma_{vac})$ is the refractive index of air for the wavelength and the APRIL, which correspond to phase and group refractive indexes, respectively. The phase refractive index $n_p(\lambda_{vac})$ is usually calculated using an empirical formula [5–8]. In the Edlén empirical equations [6–8], the phase refractive index of air can be derived from temperature, atmospheric pressure, and humidity. (Because of the limit on the length of this paper, we only considered the Edlén empirical equations in this study.) When using the Edlén equation, the default set CO_2 content is 450 ppm. When using the Ciddor equation [5], it can be set to an arbitrary value between 0 ppm and 2000 ppm. The sensitivity coefficient of CO_2 can be obtained by the procedure presented in this paper. The group refractive index of air can be estimated using [9].

$$n_g(\lambda_{cen\_vac}) = n_p(\lambda_{cen\_vac}) - \lambda_{cen\_vac} \times \left(dn_p(\lambda_{vac})/d\lambda_{vac}\right)_{\lambda_{cen\_vac}}. \tag{4}$$

Here, λ_{cen_vac} is the central frequency of the FOFC.

The following relation holds between the length in air L_{air} and the length in vacuum L_{vac}:

$$L_{air} = L_{vac}/n(\gamma_{vac}). \tag{5}$$

The uncertainty in length in vacuum due to the refractive index for data converted from the measurement in air is as follows:

$$u_n(L) = [[u(n)/n]L_{vac}. \tag{6}$$

Here, $u(x)$ denotes the uncertainty of variable x. The uncertainty of refractive index is as follows:

$$u(n) = \sqrt{K_{T0}^2 u^2(T_0) + K_{P0}^2 u^2(P_0) + K_{H0}^2 u^2(H_0)}, \tag{7}$$

where $u(T)$, $u(P)$, and $u(H)$ are the uncertainties of the instrument for measuring temperature T, barometric pressure P, and humidity H, respectively. K_T, K_P, and K_H are sensitivity coefficients and are defined as follows:

$$K_{T0} = (dn/dT)_{T0}, \quad K_{P0} = (dn/dP)_{P0}, \quad K_{H0} = (dn/dH)_{H0}, \tag{8}$$

$\left(dn_p/dT\right)_{T0}$ is the derivative of function $y = n(T, P, H)$ at $T = T_0$. The definitions are similar for $(dn/dP)_{P0}$ and $(dn/dH)_{H0}$.

Before moving to the next section, we show the difference between the above and the information given in Refs. [2,3]. In Refs. [2,3], the uncertainty in length conversion via phase refractive index was listed. In the above, we have shown a generalized expression, which can be used for the uncertainty of length conversion via not only phase refractive index but also group refractive index.

3. Numerical calculations

We used the equation for the phase refractive index given in Reference [8]. To access the program code for free, please contact the corresponding author via e-mail. The equation for the group refractive index was calculated on the basis of Eq. (4). The expressions for the sensitivity coefficient of refractive indexes were obtained by calculating the derivative of the equation of each refractive index. After obtaining an expression for the sensitivity coefficients by substituting numerical values, the values of sensitivity coefficients were calculated.

The data for a specific wavelength and specific frequencies under standard environmental conditions ($T_{std} = 20$ °C, $P_{std} = 101.325$ kPa, $H_{std} = 50\%$) are listed in Tables 1 and 2, respectively. By comparing these value with References [10–12], we confirmed the validity of the calculations.

Figs. 1–3 show the change in the sensitivity coefficients when environmental parameters change in a realistic range ($T \in [0, 40]$ °C, $P \in [60, 120]$ kPa, $H \in [5, 85]\%$). In what follows, the phase and group refractive indexes were calculated with a central wavelength of 1560.0 nm. The sensitivity coefficients K_T of both refractive indexes nonlinearly change with temperature, the sensitivity coefficients K_P of both refractive indexes linearly change with air pressure, and the sensitivity coefficients K_H of both refractive indexes are not affected by changes in humidity.

Using Eq. (6), we calculated the uncertainty in length conversion with changes in the environmental parameters, which are $u_{np(ng)}(L(T, P_{fix}, H_{fix}))$, $u_{np(ng)}(L(T_{fix}, P, H_{fix}))$, and $u_{np(ng)}(L(T_{fix}, P_{fix}, H))$. For calculation of $u_{np(ng)}(L(T, P_{fix}, H_{fix}))$, we assumed that$u(T) = 0.1$ °C, $K_{Tstr}^2 u^2(T_{str}) = K_{Pfix}^2 u^2(P_{fix}) = K_{Hfix}^2 u^2(H_{fix})$, and $T_{str} = 0$ °C. For calculation of $u_{np(ng)}(L(T_{fix}, P, H_{fix}))$, we assumed that $u(P) = 0.01$ kPa, $K_{Pstr}^2 u^2(P_{str}) = K_{Tfix}^2 u^2(T_{fix}) = K_{Hfix}^2 u^2(H_{fix})$, and $P_{str} = 60$ kPa. For calculation of $u_{np(ng)}(L(T_{fix}, P_{fix}, H))$, we assumed that $u(H) = 2\%$, $K_{Hstr}^2 u^2(H_{str}) = K_{Tfix}^2 u^2(T_{fix}) = K_{Pfix}^2 u^2(P_{fix})$, and $H_{str} = 5\%$. Subscripts "str" and "fix" denote the starting value of a variable parameter and a fixed value, respectively. Subscripts "np" and "ng" denote phase refractive index and group refractive index, respectively.L_{vac} was set to 1 m. To confirm the difference between two length conversions, we calculated the values $u_{np}(L(T, P_{fix}, H_{fix})) - u_{ng}(L(T, P_{fix}, H_{fix}))$, $u_{np}(L(T_{fix}, P, H_{fix})) - u_{ng}(L(T_{fix}, P, H_{fix}))$, and $u_{np}(L(T_{fix}, P_{fix}, H)) - u_{ng}(L(T_{fix}, P_{fix}, H))$. Figs. 4–6 show the results.

The change in the uncertainty of length conversions with temperature is not linear, whereas the changes in the uncertainty of length conversions with air pressure and the humidity are linear. A second-degree-polynomial approximation for temperature change and linear approximations for pressure and humidity changes were calculated and are listed in Table 3. The polynomial approximation error is also shown in Figs. 4–6.

It is well known that the Edlén empirical equations have an uncertainty contribution of 30–50 × 10^{-8} [8]. By comparison with this value, from Figs. 4 to 6, we can draw the following two conclusions.

First, as shown in Fig. 4, the change in uncertainties of length conversions due to a shift in temperature from 10 °C to 30 °C is about 6 nm/m, which means that by using a thermometer with an

Table 1
Sensitivity coefficients of phase refractive index under standard environmental conditions.

Wavelength (nm)	632.990577
K_T (ppb/°C)	– 0.9547
K_P (ppb/kPa)	2.684
K_H (ppm/%)	– 8.490