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Accelerating Airy beams with non-parabolic trajectories

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ABSTRACT

A class of Airy accelerating beams with non-parabolic trajectories are derived by means of a novel application of a conformal transformation originally due to Bateman. It is also shown that the salient features of these beams are very simply incorporated in a solution which is derived by applying a conventional conformal transformation together with a Galilean translation to the basic accelerating Airy beam solution of the two-dimensional paraxial equation. Motivation for the non-parabolic beam trajectories is provided and the effects of finite-energy requirements are discussed.

1. Introduction

It is well known that the two-dimensional (2D) paraxial equation

$$i\frac{\partial}{\partial s}f(\xi,s)+\frac{1}{2}\frac{\partial^2}{\partial \xi^2}f(\xi,s)=0, \tag{1}$$

written in terms of the dimensionless transverse variable $\xi=x/x_0$ and the normalized range $s=z/(kx_0^2)$, is satisfied by the expression

$$g(\xi,s)=\exp[is(s^2-6\xi)/12]Ai(\xi-s^2/4), \tag{2}$$

where $Ai(\cdot)$ denotes the Airy function. This is a variant of the infinite-energy nonspreading Airy wavepacket solution first reported by Berry and Balazs [1] in the context of quantum mechanics. One of the salient characteristic properties is the parabolic trajectory of its main lobe as it propagates along the s direction.

The first study of Airy beam solutions to Eq. (1) characterized by non-parabolic trajectories was undertaken by Torre [2,3]. Exploiting the Lie symmetry group properties of the time-dependent Schrödinger equation, she obtained the infinite-energy beam solution

$$\varphi_{Ai}(\xi,s)=\frac{1}{\sqrt{Q}}\exp\left[-i\left(\frac{1}{3Q^3}-\frac{\xi^2}{Q}+\frac{v_{\lambda_0}(\xi,s)}{Q}\right)\right]Ai[v_{\lambda_0}(\xi,s)];$$
$$v_{\lambda_0}(\xi,s)=\frac{2\xi}{Q}-\frac{1}{Q^2}-\lambda_0;$$
$$Q=2(s+a), \tag{3}$$

where λ_0 and a are free parameters. Another relevant accelerating Airy beam solution, also due to Torre, is given by

$$\Psi_{Ai}(\xi,s)=\frac{1}{\sqrt{q}}\exp\left[i\left(\frac{\xi^2}{\overline{Q}}+\frac{1}{3}S(s)^3+S\left(\frac{\xi-\delta}{q}\right)\right)\right]Ai\left(\frac{\xi-\delta}{q}\right);$$
$$S(s)=\frac{1}{4}\frac{\overline{Q}(s)}{q(s)};\ q(s)=\frac{\overline{Q}(s)}{\xi_0};\ \delta(s)=\frac{1}{4}\overline{Q}(s)S(s);\ \overline{Q}(s)=2(s+a+\xi_0), \tag{4a}$$

where ξ_0 and a are free parameters. The Airy part of this solution can be written out explicitly as follows:

$$Ai\left(\frac{\xi-\delta}{q}\right)=Ai\left[\xi_0\frac{(\xi-((\xi_0(s+a)^2)/(8(s+a+\xi_0))))}{2(s+a+\xi_0)}\right]. \tag{4b}$$

For very large values of the parameter ξ_0, the trajectory of the beam is parabolic. Deviations from a parabolic path of the beam main lobe occur for small values of ξ_0.

In a recent paper, Yan et al. [4] showed that the two-dimensional (2D) paraxial Eq. (1) has the solution

$$f(\xi,s)=\frac{1}{(1+s/a)^{1/2}}\exp[i\Phi(\xi,s)]Ai\left[2^{1/3}\frac{a(\xi-(a^3/2(s+a))-bs+a^2/2)}{s+a}\right]; \tag{5a}$$

$$\Phi(\xi,s)=\frac{u}{2(s+a)}+\alpha_s u+\frac{1}{2}\int_0^s(\alpha_s)^2dt;$$
$$u=\xi-\alpha(s);\ \alpha_s=(d/ds)\alpha(s);$$
$$\alpha(s)=\frac{a^3}{2(s+a)}+bs-\frac{a^2}{2}. \tag{5b}$$

Here, a and b are free parameters. The authors noted that "since [in the argument of the Airy function] $\alpha(s)$ is the sum of two functions: $\alpha_1(s)=a^3/[2(a+s)]-a^2/2$ and $\alpha_2(s)=bs$, the accelerating behavior of the beam is determined by the two functions. The

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functional forms of the two functions imply that at small distance of propagation, the beam has the accelerating trajectory given by the former, while, at very large distance, the accelerating is dominated by the latter." This is, essentially, the main result of the article; that the beam given in Eqs. (5a) and (5b) is characterized by a non-parabolic trajectory, compared to the basic accelerating Airy beam solution given in Eq. (2). It should be pointed out that this solution extends the Torre beam solution in Eq. (3), which preceded it. However, the method used by Yan et al. [4] can yield other types of solutions.

The goal in this article is twofold: (a) to obtain an extension of the beam solution given in Eqs. (5a) and (5b); (b) to derive a new beam solution to the 2D paraxial equation which contains several of the salient features regarding the non-parabolic nature of the trajectories. The approach toward this end will be different from the method used by Yan et al. [4]. Specifically, for case (a) a Bateman and for case (b) a conventional conformal transformation and subsequently a Galilean translation will be applied to the basic accelerating Airy solution. Since, however, both the Airy solution in Eq. (2) and the beam solution in Eqs. (5a) and (5b) contain infinite energy, the finite-energy accelerating Airy beam solution [5]

$$g_\kappa(\xi,s)=\exp[-i(2\kappa+is)(s^2-6\xi+2\kappa^2-4is\kappa)/12]Ai(\xi-s^2/4+i\kappa s), \tag{6}$$

with κ a positive parameter, will be used instead. An additional purpose for this article is to provide motivation for the non-parabolic trajectory of the beam in Eqs. (5), as well as those derived by Torre, and discuss the changes that occur when normalizability is imposed.

2. A finite-energy extension of the Yan et al. accelerating Airy beam solution

In terms of the characteristic variables $\varsigma=s-\tau$, $\eta=s+\tau$, the wavefunction

$$u(\xi,\varsigma,\eta)=\exp(i\eta/2)g_\kappa(\xi,\varsigma) \tag{7}$$

obeys the non-dimensionalized homogeneous scalar wave equation

$$\left(\frac{\partial^2}{\partial\xi^2}+4\frac{\partial}{\partial\varsigma\partial\eta}\right)u(\xi,\varsigma,\eta)=0. \tag{8}$$

In 1910, Bateman [6,7] discovered a transformation, more general than a conformal change of the metric, which could be used to transform solutions of Maxwell's equations into similar ones. In the case of the scalar wave equation, in particular, the Bateman transformation assumes the form

$$u_1(\xi,\varsigma,\eta)=\frac{1}{\sqrt{\varsigma}}u\left[\frac{\xi}{\varsigma},-\frac{A}{\varsigma},\frac{\xi^2+\varsigma\eta}{2A\varsigma}\right], \tag{9}$$

with A a free parameter. The function $u_1(\xi,\varsigma,\eta)$ also obeys the scalar wave Eq. (8). It is not well known that this new wavefunction leads to a new solution of the 2D paraxial Eq. (1) as follows:

$$\phi(\xi,s)=u_1(\xi,s/A,0). \tag{10}$$

In order to achieve the goal of extending the beam solution in Eqs. (5a) and (5b), a conventional Galilean transformation is carried out next; specifically,

$$f_e(\xi,s)=\exp\left[i\frac{v}{2}\left(\xi-\frac{v}{2}\frac{s}{2}\right)\right]\phi\left(\xi-v\frac{s}{2},s\right), \tag{11}$$

where v is a free parameter. With the additional translations $s\to s+a$, $\xi\to\xi+a^2/2$, the extended solution $f_e(\xi,s)$ can be written out explicitly as follows:

$$f_e(\xi,s)=\frac{1}{(1+s/a)^{1/2}}\exp[i\Phi_e^{(1)}(\xi,s)]\exp[-\Phi_e^{(2)}(\xi,s)]\times Ai\left[\frac{A(\xi-A^3/(4(s+a))-(v/2)s+(a^2/2)-iA\kappa)}{s+a}\right]; \tag{12a}$$

$$\Phi_e^{(1)}(\xi,s)=\frac{1}{12}\left[3v\phi_1+\frac{A^2(A^4+(s+a)(-3A\phi_2+((3(s+a)\phi_2^2)/2A^2)))}{(s+a)^3}\right];$$

$$\Phi_e^{(2)}(\xi,s)=\frac{\kappa\left[3A^2-3A(s+a)\phi_2+(s+a)^2\kappa^2\right]}{6(s+a)^3}+i\frac{A^2\kappa}{2(s+a)^2};$$

$$\phi_1=a^2-\frac{vs}{2}+2\xi,\ \phi_2=a^2-vs+2\xi. \tag{12b}$$

The factor $\Phi_e^{(2)}(\xi,s)$ in the second exponent in Eq. (12a) is due solely to the incorporation of finite energy in the beam. A comparison of the arguments of the Airy functions in Eqs. (5) and (12a) shows that $f_e(\xi,s)$ reduces to $f(\xi,s)$ if $A=2^{1/3}a$ and $v=2b$ in the case of infinite energy beams ($\kappa=0$). The procedure leading to the extended beam solution in Eqs. (12a) and (12b) makes clear that the terms $A^3(s+a)^{-1}/4$ and $vs/2$ in Eq. (12a) and, in particular, $a^3(s+a)^{-1}/2$ and bs in Eqs. (5a) and (5b) are due to a conformal transformation and a Galilean translation, respectively.

The effects of finite energy present in the beam solution $f_e(\xi,s)$ are illustrated in Fig. 1 below, where a comparison is made with the beam solution in Eq. (5).

The parameters A and v in Eqs. (12a) and (12b) have been chosen so that the two beam solutions are identical for $\kappa=0$. A plot of $|f(\xi,s)|^2$ is shown in Fig. 1a. The main lobe of the beam follows a non-parabolic path, its amplitude decreases monotonically as the range s increases, and its width increases. Fig. 1b shows a plot of $|f_e(\xi,s)|^2$ with $\kappa=2\times10^{-2}$. The main lobe of the beam follows a non-parabolic trajectory which essentially is identical to that shown in Fig. 1a. However, a focusing effect is exhibited in this

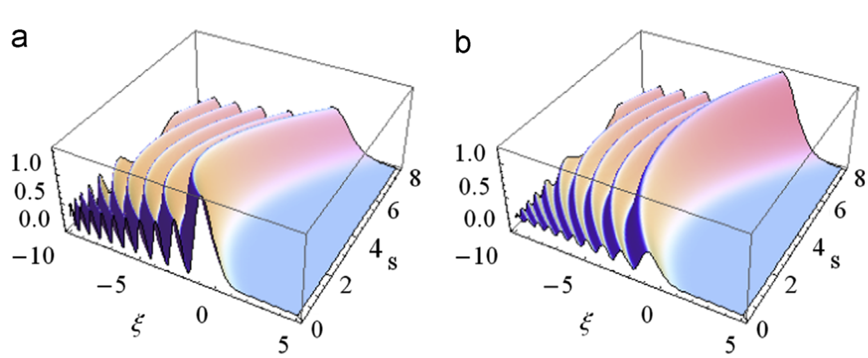

Fig. 1. (a) $|f(\xi,s)|^2$ with $a=10$ and $b=3$; (b) $|f_e(\xi,s)|^2$ with $a=10$, $A=2^{1/3}a$, $v=6$ and $\kappa=2\times10^{-2}$.