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Parametric competition in non-autonomous Hamiltonian systems

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ABSTRACT

In this work we use the formalism of chord functions (*i.e.* characteristic functions) to analytically solve quadratic non-autonomous Hamiltonians coupled to a reservoir composed by an infinity set of oscillators, with Gaussian initial state. We analytically obtain a solution for the characteristic function under dissipation, and therefore for the determinant of the covariance matrix and the von Neumann entropy, where the latter is the physical quantity of interest. We study in details two examples that are known to show dynamical squeezing and instability effects: the inverted harmonic oscillator and an oscillator with time dependent frequency. We show that it will appear in both cases a clear competition between instability and dissipation. If the dissipation is small when compared to the instability, the squeezing generation is dominant and one can see an increasing in the von Neumann entropy. When the dissipation is large enough, the dynamical squeezing generation in one of the quadratures is retained, thence the growth in the von Neumann entropy is contained.

1. Introduction

The dynamics of quantum open systems has raised increasing interest of physicists specially in the last decades [1,2]: it can be directly connected to the non-observance of quantum phenomena in the classical world. The same phenomena which led Schroedinger to discredit his own theory is directly connected to the linear structure of the Hilbert space. Most of them have nowadays been observed and the usual approach as to why they are not present in our everyday life is to consider that quantum mechanics was first conceived for closed systems and effects of the surrounding environment, when included, tend to wash out quantum properties [1].

This problem is however far from being a closed issue and the classical limit of quantum mechanics is still a matter of enthusiastic debates [1–4]. In particular, the questions on quantum-to-classical transition acquire a singular aspect in the case of quantum systems with nonlinear or chaotic classical counterparts. If dissipation is absent, it is expected that instabilities yield the fast spreading of the wave function throughout the phase space for such systems, especially the macroscopic ones. Thus, an initially well localized wave packet will soon be fragmented throughout available regions of the phase space, and coherent superpositions will appear between the fragments, leading to a rapid breakdown of the correspondence between classical and quantum descriptions. Some authors [5–9] advocate that the unavoidable interaction of a macroscopic system with its environment is essential to prevent the appearing of these quantum signatures yielded by inherent instabilities exhibited by the unitary evolution. Notwithstanding, other authors [10] sustain that the coupling with an environment is not necessary because such quantum effects are so tiny that they are not measurable, especially in the case of macroscopic objects. This controversy only stresses the importance of the study of the role played by instabilities in the question of quantum-to-classical transition.

In the present contribution, we are concerned to the questions: what happens if the unitary evolution, *i.e.* the Hamiltonian of the problem, may lead to instability? What role this instability effects does play? Examples of application of non-autonomous Hamiltonian systems can be found in a huge range of areas of physics, in particular: in quantum optics, where a harmonic oscillator with time dependent frequency is shown to generate squeezing [11,12], tunneling [13], exact solutions for mathematical problems and toy models [14], parametric amplification [15], quantum Brownian motion [16]. Most of these works employs the model of the harmonic oscillator with time dependent frequency. It is worth to mention that this model is largely studied both in classical and quantum physics and, as a merit, is amenable to analytical treatment. In fact, the time independent Schroedinger equation for the harmonic oscillator with time dependent frequency assumes the form of Hill differential equation, which, in turn, is a particular form of Pinney equation. Examples of Hill or Pinney equation in physics can be found in studies on synchrotron accelerators [17], anisotropic

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Bose–Einstein condensates [18,19], Paul traps [20], and cosmological models of particle creation [21]. Further, one of the first approaches to include dissipation in quantum physics employed a class of time dependent Hamiltonians, known as Caldirola–Kanai Hamiltonians [16,22]. Even in cosmology, in the inflationary era, when quantum effects are supposedly important, studies using non-autonomous Hamiltonians, leading to instabilities and squeezing effects are found [23]. One then frequently uses non-autonomous unitary evolutions of the same type, now modeling transitions between harmonic oscillators which give rise to particle formation [11]. Quantum chaos and instabilities also arise in recent experiments and theoretical models [24], rendering new perspectives to this interesting area in physics. It is interesting to note that, due the features shared by both models, some authors propose the Bose–Einstein condensates as a test bench of some cosmological scenarios [19].

Another interesting problem was raised by Zurek and coworkers [25] as to the rate of entropy increase when the system of interest is coupled not to a reservoir but to an unstable, two degrees of freedom system. In Ref. [14] the authors analytically showed that, in fact, entropy grows faster, but for that, chaos is not necessary (although sufficient). Instability alone already reflects this physics. Also, more realistically, as discussed in [14], the potential modelling Paul–Penning traps [26] has instability points which can be, to a certain degree, approximated by an inverted oscillator. What happens to the well known physics described, if an environment is added to the non-autonomous unitary dynamics? Can dissipation stop the inevitable acceleration caused by instabilities?

A word about the formal mathematical approach to the problem is in order: for autonomous systems, there are several possibilities to solve a master equation. One of the frequently used and powerful tools is that of Lie algebras of superoperators. Perhaps that is the reason why there is not so much work devoted to the question of non-autonomous systems evolving under nonunitary dynamics. As discussed above, however, several interesting issues may be cleared, if one manages to formulate the problem in appropriate language. In the present case, we will be considering single-mode Gaussian states. For these states, all we need are the second statistical moments or the covariance matrix, which can be gotten very simply as derivatives of the characteristic function (the Fourier transform of the Wigner function), by taking the derivatives of this function at the origin [27,28]. Moreover, a very elegant theoretical method for Wigner functions and nonunitary quadratic evolutions is given in Ref. [28]. It involves several classical elements, rendering the physics of the problem very transparent and the inclusion of nonunitary terms is natural.

In Section 2 we present an analytical solution for the characteristic function, using the most general bilinear Lindbladian (for dissipative reservoirs). We show our results for the inverted harmonic oscillator (IHO) and for a non-autonomous harmonic oscillator (NAHO) with frequency $\omega(t)=\omega_0\sqrt{1+\gamma t}$ in Section 3 and in the last section we make our final remarks.

2. Analytic solution for the Wigner and characteristic function

In this section we review some aspects concerning the evolution of single-mode Gaussian states under dissipation. The literature is plenty of references on this subject (theory and applications) [11,27–36]. To obtain our main result — analytical solutions for non-autonomous Hamiltonians — this section is, although straightforward, useful.

2.1. Unitary dynamics of single mode Gaussian states

We can define a general form of the Hamiltonian part of the equations of motion for both models studied in this work, namely, the inverted harmonic oscillator (IHO) and the non-autonomous harmonic oscillator. The Hamiltonian reads

$$\hat{H}(\hat{q},\hat{p},t)=\frac{\hat{p}}{2\,m}+\frac{1}{2}m\omega^2(t)\hat{q}^2, \tag{1}$$

where $\hat{q}$ and $\hat{p}$ are position and linear momentum operators, respectively, m is the mass of the oscillator and $\omega(t)$ is a time-dependent frequency. If we take $\omega_0=|\omega(0)|$, the annihilation and creation operators for $t=0$, $\hat{a}$ and $\hat{a}^\dagger$, are given by $\hat{a}=\sqrt{m\omega_0/2\hbar}(\hat{q}+i(\hat{p}/m\omega_0))$ and $\hat{a}^\dagger=\sqrt{m\omega_0/2\hbar}(\hat{q}-i(\hat{p}/m\omega_0))$. The Hamiltonian above can be written as [35,36]

$$\hat{H}(\hat{a},\hat{a}^\dagger,t)=\hbar\left[f_1(t)\left(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}\right)+f_2(t)(\hat{a}^{\dagger 2}+\hat{a}^2)\right], \tag{2}$$

where $f_1(t)=\omega_0/2[(\omega(t)/\omega_0)^2+1]$ and $f_2(t)=\omega_0/4[(\omega(t)/\omega_0)^2-1]$.

In order to establish the notation, we will first present single-mode Gaussian states and its parameters, well known in the literature by several methods. The initial state is

$$\hat{\rho}(0)=\hat{D}(\alpha(0))\hat{S}(r(0),\phi(0))\hat{\rho}(\nu(0))\hat{S}^\dagger(r(0),\phi(0))\hat{D}^\dagger(\alpha(0)), \tag{3}$$

where all the parameters are given by the first and second moments:

$$\begin{aligned}
\alpha&=\langle\hat{a}\rangle\\
\alpha^*&=\langle\hat{a}^\dagger\rangle\\
e^{i\phi}&=\sqrt{\frac{\sigma_{\hat{a}\hat{a}}}{\sigma_{\hat{a}^\dagger\hat{a}^\dagger}}}\\
\nu&=\sqrt{\left(\sigma_{\hat{a}^\dagger\hat{a}}-\frac{1}{2}\right)^2-\sigma_{\hat{a}^\dagger\hat{a}^\dagger}\sigma_{\hat{a}\hat{a}}}-\frac{1}{2}\\
r&=\frac{1}{4}\ln\left(\frac{\sigma_{\hat{a}^\dagger\hat{a}}-\frac{1}{2}+\sqrt{\sigma_{\hat{a}^\dagger\hat{a}^\dagger}\sigma_{\hat{a}\hat{a}}}}{\sigma_{\hat{a}^\dagger\hat{a}}-\frac{1}{2}-\sqrt{\sigma_{\hat{a}^\dagger\hat{a}^\dagger}\sigma_{\hat{a}\hat{a}}}}\right).
\end{aligned} \tag{4}$$

In the equations above $\sigma_{\hat{a}^\dagger\hat{a}^\dagger}=\langle(\hat{a}^\dagger)^2\rangle-\langle\hat{a}^\dagger\rangle^2$, $\sigma_{\hat{a}\hat{a}}=\langle(\hat{a})^2\rangle-\langle\hat{a}\rangle^2$, $\sigma_{\hat{a}^\dagger\hat{a}}=\langle\hat{a}^\dagger\hat{a}\rangle-\langle\hat{a}^\dagger\rangle\langle\hat{a}^\dagger\rangle+1$. Those parameters are related to displacement (α), squeezing (r,ϕ) and "impurity" (ν) of the state. In our study the initial state will always be in this general single-mode Gaussian form and, since the dynamics is quadratic, the state will evolve as a single-mode *Gaussian* state [28].

One can study the state by analyzing the evolution of the parameters above, or the covariance matrix (CM):

$$\sigma=\begin{pmatrix}\langle\hat{q}^2\rangle-\langle\hat{q}\rangle^2 & \frac{1}{2}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle-\langle\hat{q}\rangle\langle\hat{p}\rangle\\ \frac{1}{2}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle-\langle\hat{q}\rangle\langle\hat{p}\rangle & \langle\hat{p}^2\rangle-\langle\hat{p}\rangle^2\end{pmatrix}. \tag{5}$$

2.2. Wigner and characteristic functions — dissipationless case

The Wigner function is defined as [32]

$$W(\vec{x})=\frac{1}{2\pi\hbar}\int dq'\left\langle q+\frac{q'}{2}\middle|\hat{\rho}\middle|q-\frac{q'}{2}\right\rangle\exp\left(-i\frac{pq'}{\hbar}\right), \tag{6}$$

where $\vec{x}=(p,q)$. It propagates "classically" for up to quadratic dynamics [28]:

$$\frac{\partial}{\partial t}W_t(\vec{x})=\left\{H(\vec{x}),W_t(\vec{x})\right\}, \tag{7}$$

where $\{f,g\}=(\partial f/\partial q)(\partial g/\partial p)-(\partial f/\partial p)(\partial g/\partial q)$ is the classical Poisson bracket, and $H(\vec{x})=\vec{x}\cdot\hat{H}\vec{x}$.

One can write the propagated Wigner functions as [37]

$$W_t(\vec{x})=W_0(\mathbf{R}_{-t}\vec{x}), \tag{8}$$

where

$$\mathbf{R}_t=\exp(2\mathbf{\Omega}\hat{H}t), \tag{9}$$