Optics Communications 313 (2014) 430–435

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Efficient sensitivity analysis approach based on finite element solutions of photonic structures

Mohamed R. Abdelhafez, Mohamed A. Swillam *

Department of Physics, The American University in Cairo, P.O. Box 74 New Cairo 11835, Egypt

ARTICLE INFO

Article history:
Received 13 April 2013
Received in revised form
4 September 2013
Accepted 9 October 2013
Available online 1 November 2013

Keywords:
Finite Element Method
Sensitivity analysis
Modal analysis
Adjoint variable method

ABSTRACT

We propose, for the first time, an efficient technique for the sensitivity analysis of photonic waveguide structures with arbitrary shapes. Those waveguides are analyzed using the Finite Element Method (FEM). Our technique uses the solution obtained by any FEM solver and without the need of performing any extra FEM simulation, it gives accurate results for the sensitivity of the modal parameters with respect to all the design parameters. This technique is far more efficient than the traditional numerical estimates of the sensitivities.

1. Introduction

The Finite Element Method (FEM) is favoured in solving partial differential equations especially in complex structures because it allows for flexible meshes depending on the shape of the structure. FEM gives more accurate results than the traditional Finite Difference Methods (FDM) in extracting the electric modal profile of optical waveguides and their respective propagation parameters. In general, optical devices have many design parameters affecting their modal parameters such as the refractive indices of their materials and the dimensions of their subregions. It is desired not only to solve for the solution in a certain arrangement but also to numerically calculate the sensitivity of the solution with respect to changes in the design parameters. Obtaining the sensitivity of the solution to design parameters is essential in calculating some key functional parameters such as the propagation loss of a waveguide, the beating length of a directional coupler, the length of polarization converters and the imaging length of multimode interference devices. They are also important for the manufacturing process in order to calculate the yield analysis and tolerance. Recently, sensitivity analysis for finite-difference-based solutions has been introduced in photonics with good accuracy [1,2]. These sensitivity approaches are later utilized to engineering the dispersion characteristics of photonic devices [3]. In addition, sensitivity analysis of microwave structures has been introduced for both finite difference [3] and finite element approaches. [4–8]. In general, no previous work has been done on sensitivities of FEM solutions in photonics. In particular, sensitivity analysis using FEM for modal analysis has never been presented before for photonics applications.

In general, there is no analytical approach for estimating the sensitivities of any response that is calculated using numerical technique either FD or FEM. Thus, usually a numerical approximation using FDM is utilized for estimating the sensitivities numerically. However, this approach requires one extra simulation for each design parameter if the Forward or Backward FDM is used. If a higher accuracy is desired, central FDM is used but it requires two extra simulations. Therefore, for a structure with N design parameters, to get accurate results for the sensitivity using FDM techniques, we need extra 2N simulations which is highly inefficient special for structures that is solved using FEM which requires huge resources at each simulation.

In this paper, we present an efficient scheme for calculating the sensitivities of FEM solutions which does not require performing any extra FEM simulation. A preliminary illustration of this approach was given in [9]. In this paper, a detailed analysis and implementation of this approach are given here. In addition few more examples that illustrate the universality of the approach is also included in this paper. We start by proposing the theory of our technique in Section 2. Then, we show the results of using this technique in several waveguide examples in Section 3. Finally, the conclusion of our work is presented in Section 4.

2. Theory and implementation

2.1. The wave equation as an Eigenvalue problem

The Wave equation in the Transverse Electric (TE) mode of the waveguide can be written as

$$\nabla \times \nabla \times E - k_0^2 \varepsilon_r E = 0 \quad (1)$$

* Corresponding author at: Department of Physics, The American University in Cairo, P.O. Box 74 New Cairo 11835, Egypt.
E-mail address: m.swillam@aucegypt.edu (M.A. Swillam).

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http://dx.doi.org/10.1016/j.optcom.2013.10.031

where

$$k_0 = \frac{2\pi}{\lambda}, \tag{2}$$

where λ is the wavelength of the wave in vacuum, ε_r is the relative permittivity of the medium and E is the vector of transverse electric fields. Using FEM, the waveguide can be thought of as being meshed into triangles and the equation is solved locally in the three nodes of each triangle. Therefore, the wave equation can be transformed into a matrix operator form applied at the nodes of the mesh [10].

$$KE = \beta^2 ME \tag{3}$$

where K and M are operator matrices, E is the transverse eigenmode in the waveguide and β is the corresponding eigenvalue of the propagation constant. This can be written as

$$AE = \beta^2 E \tag{4}$$

where

$$A = M^{-1}K \tag{5}$$

Differentiating (4) with respect to the design parameter p_i (where i is the parameter index $1, 2, \ldots, N$), we get

$$\frac{\partial A}{\partial p_i}E + A\frac{\partial E}{\partial p_i} = 2\beta\frac{\partial \beta}{\partial p_i}E + \beta^2\frac{\partial E}{\partial p_i} \tag{6}$$

Multiplying both sides by the Hermitian conjugate of the left Eigenfunctions E_L, we get

$$E_L^H\frac{\partial A}{\partial p_i}E + E_L^H A\frac{\partial E}{\partial p_i} = 2\beta E_L^H\frac{\partial \beta}{\partial p_i}E + \beta^2 E_L^H\frac{\partial E}{\partial p_i} \tag{7}$$

Rearranging the terms gives us the expression for the sensitivity of β [11].

$$\frac{\partial \beta}{\partial p_i} = \frac{1}{2\beta}\frac{E_L^H\frac{\partial A}{\partial p_i}E}{E_L^H E} \tag{8}$$

Furthermore, for scalar wave analysis, the system matrix is symmetric. This condition also holds for the semivectorial and vectorial cases if the structures are weakly guided. For the semivectroial case, our experience shows that the system matrix is nearly symmetric. Therefore, assuming A is Hermitian for both scalar and semivectorial case, we get

$$E_L = E \tag{9}$$

and the expression simplifies to

$$\frac{\partial \beta}{\partial p_i} = \frac{1}{2\beta}\frac{E^H\frac{\partial A}{\partial p_i}E}{E^H E} \tag{10}$$

It is sometimes more convenient to normalize β by the wavenumber k_0 to get the effective refractive index of the waveguide

$$n_{eff} = \frac{\beta}{k_0} \tag{11}$$

Therefore, using (10) and (11), the expression for the sensitivity of n_{eff} is

$$\frac{\partial n_{eff}}{\partial p_i} = \frac{1}{2\beta k_0}\frac{E^H\frac{\partial A}{\partial p_i}E}{E^H E} \tag{12}$$

The expression shows that to get the sensitivity of n_{eff}, we only need to have the derivative of the matrix A of the already done simulation. Therefore, if we know the sensitivity of A with respect to the wanted design parameter, no extra simulations are needed compared to 2 extra simulations per parameter needed using central finite difference schemes.

The derivative of A may be analytically available if the matrices K and M are analytically differentiable with respect to the design parameter. Then, we can utilize the chain derivative rule of (5) to get

$$\frac{\partial A}{\partial p_i} = -M^{-1}\frac{\partial M}{\partial p_i}M^{-1}K + M^{-1}\frac{\partial K}{\partial p_i} \tag{13}$$

In case K and M do not have analytical derivatives, then we utilize perturbation theory to calculate their derivatives using the central finite difference method (CFDM). The matrix A is obtained from the stiffness matrices K and M using LU factorization. This process is performed in order to obtain the original solution and readily utilized for obtaining the sensitivity. We use CFDM to regenerate the sensitivity of the matrix A with respect to any design parameter p_i around the original point but without solving the eigenvalue system again. The derivative of matrix A is then given by

$$\frac{\partial A}{\partial p_i} = \frac{A(p_i + \Delta p_i) - A(p_i - \Delta p_i)}{2\Delta p_i} \tag{14}$$

2.2. The AVM method

Our technique in estimating the sensitivity of FEM solutions is called the Adjoint Variable Method (AVM) because it solves for the sensitivity as an adjoint problem that is independent of the solver of the initial problem. As obvious from (12), to perform the sensitivity analysis of the original problem, the only parameters needed are the FEM solution and its system matrix derivatives (E, β, $\partial A/\partial p_i$). both E and β are already supplied by the FEM solver. One advantage of the AVM, other than speed and efficiency, is not requiring the FEM solver to solve the problem using a specific algorithm. The solver can use either node or edge analysis and can use any of the FEM different methods (Galerkin, Least Squares, etc.). The AVM then uses the resulting E and β regardless of the method of the solver. The system matrix (A) is then either supplied by the solver (if the solver uses a relevant technique and allows for making it available for users) or generated by the AVM using a designed mesh that transforms the solution into the matrices of the form of (3). The mesh is then perturbed to generate the matrices $A(p_i \pm \Delta p_i)$ and then calculate the matrix derivative according to (14). In addition, the FEM solver can use the AVM to extend its capabilities to include calculating the sensitivities as part of the simulation using the analytic derivatives of their system matrices as in (13) or by perturbing the mesh as in (14). The implementation steps are illustrated in a flowchart diagram in Fig. 1. In this flowchart, the main steps are: (1) the FEM of the structure is solved using any FEM approach, in our case Glarkin method for the Semivectorial case is implemented, (2) once the system equation A is obtained from the original solution, the design parameters are defined, (3) the dependant of the design parameter on the system matrices is then examined order to calculate the derivative of the system equation A. For analytical dependance, the derivative of the matrix A can be obtained analytically. Examples for such dependence include the refractive index dependance which allow for obtaining analytical derivative of the system matrices with respect to the change of the refractive index change. A simpler approach is done by using perturbation theory to obtain simple expression for the sensitivity as given in (14) and (4) once the eigen values and eigen vectors are obtained from step (1) and the derivative is obtained form step (4), the sensitivity of the design parameters can be directly obtained from (10) and (12). Finally, steps 3 and 4 are repeated till the sensitivity of all the design parameters are obtained.