Hamiltonian and phase-space representation of spatial solitons

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ABSTRACT

We use Hamiltonian ray tracing and phase-space representation to describe the propagation of a single spatial soliton and soliton collisions in a Kerr nonlinear medium. Hamiltonian ray tracing is applied using the iterative nonlinear beam propagation method, which allows taking both wave effects and Kerr nonlinearity into consideration. Energy evolution within a single spatial soliton and the exchange of energy when two solitons collide are interpreted intuitively by ray trajectories and geometrical shearing of the Wigner distribution functions.

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1. Introduction

Spatial solitons, where optical beams travel without divergence or convergence in a nonlinear medium, have been theoretically presented and experimentally demonstrated in various physical systems [1–6]. Many applications have been proposed for solitons and their interactions, including optical-fiber communication systems [7], "gateless" computers [8], and soliton navigation [9]. Although theoretical methods, e.g. inverse scattering theory, exist for a few special cases [10], in others, it is not an easy task to predict soliton's behavior. Thus, one has to use numerical methods, such as split-step Fourier method [11]. In particular, energy exchange during soliton collisions has been under extensive research [10,12], but much remains unknown, especially the detailed evolution of power flow during collisions. In this paper, we present a novel perspective on the propagation of spatial solitons and the energy interactions during multi-soliton collision, using ray tracing and phase-space representations.

Since rays represent power flows, ray diagrams are physically intuitive and provide useful insights for the evolution of energy during a nonlinear optical phenomenon. In addition, ray tracing is easy to interpret with traditional optical terms such as ray-intercept plots and aberrations. As a result, we expect ray description to be highly beneficial for understanding complex nonlinear phenomena. However, traditional ray tracing method

cannot take wave effects such as diffraction and interference into consideration [13]. Furthermore, solving ray-tracing equations in Kerr nonlinear media is not straightforward because of the coupling between optical intensity and refractive index. In this paper, we propose to calculate ray trajectories using the iterative nonlinear beam propagation method [14]. This method provides a rigorous way to include both wave effects and nonlinearity into the ray-tracing results. Wave effects are considered by applying the Wigner distribution function (WDF) to Hamiltonian ray tracing as the initial condition of the rays. Kerr nonlinearity, where the refractive index changes according to the local optical intensity [15], is included by an iterative process which updates the refractive index and intensity profiles at each iteration. The WDF [16–18] is a phase-space representation of the coherence property of an optical beam. It defines a generalized ray picture, known as the generalized radiance, which is a function of position and momentum [19]. Along each ray, the radiance is conserved [18]. The optical intensity at any point of space can be calculated from the WDF through a projection along the momentum direction. The iterative nonlinear beam propagation method has been previously shown as a versatile tool for the design of nonlinear optical devices [20]. Here we show that the same method can provide useful physical insight of spatial soliton's propagation, collision and evolution with the use of ray diagrams and rigorous consideration of wave effects through the WDF.

In this paper, Hamiltonian ray diagrams and phase-space representations of spatial solitons and multi-soliton collisions are studied. Energy evolution is discussed through the spatial trajectories of rays. Here, only the propagation of a single spatial soliton and the collision of two solitons are shown as examples; the same

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ray tracing and phase-space representation approach can be easily applied to other complex nonlinear phenomena, especially those subject to the third order nonlinear susceptibility, such as self-focusing, laser beam breakup (filamentation), soliton breathers, and dark solitons. Furthermore, ray representation mentioned in this paper may also be applied to the study of temporal nonlinear phenomena. For example, since spatial propagation of spatial solitons is analogous to temporal evolution of temporal solitons through a direct mapping between the space and time variables, ray tracing results presented here may be extended to temporal solitons by straightforward modifications. Note that among all the possible scenarios, our iterative method gives a good approximation when the rays are paraxial and the refractive index distribution does not have rapid variation. Otherwise the transport model of the WDF may fail [18]. In those cases, errors will build up especially within regions of sharp index or intensity variations, e.g. a focal point.

2. Spatial soliton description

To investigate the Hamiltonian properties of a spatial soliton, we first show that given the known refractive index profile of the nonlinear medium where the soliton propagates, Hamiltonian ray trajectories yield a self-consistent result. More specifically, we show that at any given plane transverse to the optical axis, all rays have traveled the same optical path length (OPL); moreover, the intensity distribution $I(x)$ [and thus the index profile according to the Kerr effect relation $n(x) = n_0 + n_2 I(x)$] maintains the same profile.

Hamiltonian equations describe a ray trajectory by its position x and momentum p_x along x direction at any transverse plane z, for a given index distribution $n(x,z)$, and can be written as [21]

$$\frac{dx}{dz} = \frac{\partial h}{\partial p_x} = -\frac{p_x}{h}, \quad \frac{dp_x}{dz} = -\frac{\partial h}{\partial x} = -\frac{n}{h}\frac{\partial n}{\partial x}, \tag{1}$$

where $h = -\sqrt{n^2 - p_x^2}$ is the screen Hamiltonian. Note that the momentum is proportional to the direction of ray propagation by $p_x = \sin\phi/\lambda$, where ϕ is the angle of the propagation with respect to z-axis, and λ is the wavelength. Based on the nonlinear Schrödinger equation, there exists an analytical solution for the optical field of a spatial soliton:

$$A(x,z) = A_0 \operatorname{sech}(x/w_0) \exp(i\theta(z)), \tag{2}$$

where A_0 is the peak amplitude, w_0 is the beam width and θ is the phase which is invariant along x direction [22]. In a Kerr nonlinear medium, the refractive index changes proportional to the intensity distribution; thus the index profile for the spatial soliton is

$$n(x,z) = n_0 + n_2 A_0^2 \operatorname{sech}^2(x/w_0), \tag{3}$$

where n_0 is the usual, weak-field refractive index, and n_2 is the Kerr effect coefficient. Given the index distribution, the ray trajectories can be obtained by solving the pair of Hamiltonian equations in Eq. (1). To obtain an input ray distribution consistent to the field description in Eq. (2), we compute the WDF $\mathcal{W}(x,p_x)$ of $A(x,z=0)$ to define the initial rays for Eq. (1), according to the definition [18,23]

$$\mathcal{W}(x,p_x) = \int A\left(x + \frac{x'}{2}\right) A^*\left(x - \frac{x'}{2}\right) e^{-ip_x x'}\, dx'. \tag{4}$$

In the simulation shown in Fig. 1(a), we used $n_0 = 1.5$, $n_2 = 2 \times 10^{-13}$ (m/V)^2, $A_0 = 281$ V/m and $w_0 = 0.55$ mm. Note that although each ray takes a distinct periodic trajectory, the rays propagate around the central region of the soliton experience higher refractive indices; the total OPLs [defined as the path integral of $n(x,z)$ along a ray trace] of all the rays at any transverse plane are the same.
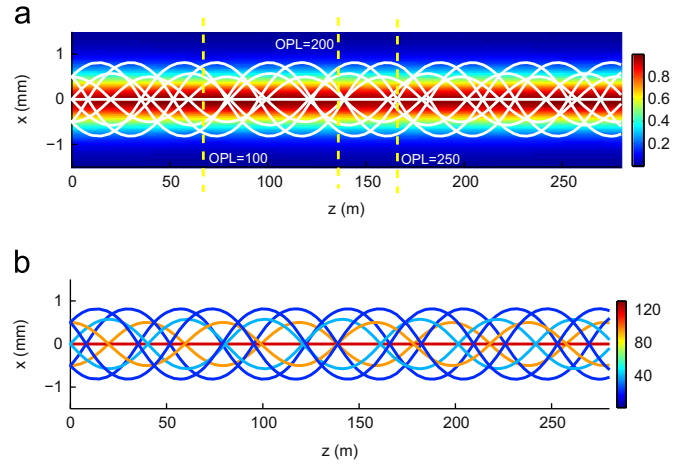
Fig. 1. (a) Hamiltonian ray tracing results based on the known index profile of a spatial soliton, and (b) iterative nonlinear beam propagation method results starting from a medium of constant weak-field index. Solid lines are a subset of all 10,100 rays used in the simulation. In (a), color shading denotes the distribution of the normalized intensity profile, proportional to the nonlinear index change $n_2 I(x)$. Dashed vertical lines indicate the wavefronts with respect to different OPLs. In (b), colors of lines indicate the generalized radiances carried by these rays. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

This result suggests that the wavefronts are always perpendicular to the optical axis which agrees with the definition of θ in Eq. (2).

Next we consider a dynamic process where the initial refractive index is a constant n_0, and show that given the initial rays satisfying the fundamental soliton solution [in Eq. (2)] at the input plane of a Kerr medium, the solution to the Hamiltonian equations converges to the same index distribution as Eq. (3). We demonstrate this result using our iterative nonlinear beam propagation method. The method starts with a medium of constant weak-field refractive index n_0, and the definition of all the initial rays, i.e. initial position and direction, based on the WDF of the input "sech" profile. Each ray carries a generalized radiance, given by the value of WDF at the given position and momentum. At each iteration, we apply Hamiltonian ray tracing for each ray for the current index distribution; at the end of each iteration, the intensity at each point of space is calculated as the sum of the generalized radiances carried by all the rays passing through the point, according to the projection property of the WDF. Refractive index distribution is then updated according to the Kerr effect, whose result is used in the next iteration. As the iterations continue, all the rays converge to form a soliton. Total simulation time is 3 min 7 s. The converged ray trajectories are shown in Fig. 1(b), which match the result in Fig. 1(a). Here the intensity profile [i.e. the refractive index profile according to $n(x) = n_0 + n_2 I(x)$] is the same as Fig. 1(a), thus we are not showing it again. Note that this single soliton propagation result has been presented in [14] as an example of the iterative nonlinear beam propagation method. In this section we perform detailed analyses on this result and verify it with analytical expression of a spatial soliton in Eq. (2).

According to the ray tracing results, rays with different generalized radiances and initial condition oscillate at different periods. Though most of the rays propagate in oscillatory fashion instead of straight lines parallel to the optical axis, the generalized radiances of all rays sum up to the correct intensity profile of a spatial soliton. The WDFs calculated from the rays intercepting two different z planes are shown in Fig. 2. As illustrated in the figure, both the WDF and the intensity distribution remain invariant as the soliton propagates, which matches the Hamiltonian ray tracing description and also the analytical results. To further illustrate this point, here we also include an animation showing the WDFs and sampled rays at different z