



Mutual space–frequency distribution: Analytic calculations of Gaussian signal

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ABSTRACT

Mutual space–frequency distribution is proposed and both the Wigner and Weyl distribution functions are shown to be some particular cases of this distribution. The simple connection is established between the Wigner and Weyl distributions. Mutual distribution of Gaussian signal is analytically obtained. It is shown that the Wigner distribution is formed as the rotational displacement of the Weyl distribution on informational diagram of conjugate coordinates $(x; p)$ on an angle proportional to the mutual parameter t . The results are presented of direct calculations of mutual distribution for Gaussian signal in the mutual domain.

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1. Introduction

The main aim of space–frequency analysis is elaboration of distributions in order to get the information about signal simultaneously in coordinate and frequency domains. As a rule the Fourier transform is used to receive the signal frequency spectrum. This well-known transformation is a good tool for the analysis of signal intensity distribution in the frequency domain. Such analysis foresees the calculation of Fourier-spectrum for constant coordinates. Practically, we have to deal with a certain momentary value of coordinates for which the signal and their Fourier-image are simultaneously determined. With the coordinate variation the appropriate conversion of Fourier-spectrum also takes place and at once the problem of signal analysis occurs; the latter contains frequency components that are variables in accordance with coordinates. In such a case it is important to know the value of coordinate at which the corresponding transformation of frequency spectrum takes place. In order to investigate the variation of signal spectrum with the variation of its coordinate, as far back as in 1960–1980s a new approach was propounded. It unites the information about coordinate and frequency constituents of the signal in the so-called space–frequency representations. In such representations a certain mutual function of coordinate and frequency is considered. The idea of constructing mutual representations originates in works of Wigner [1], Gabor [2] and Weyl [3]. Before the 1980s of the previous century tens of space–frequency representations of this

kind were taken under consideration [4–6]. However, the Wigner and Weyl distributions, which are most used for the present day, remained the prerogative of quantum mechanics and they have not had precisely expressed use. Clasen and Mecklenbrauker worked out the theory of application of Wigner distribution for the space–frequency analysis of signals. Its main results were published in the series of works under the title “Wigner distribution – the instrument for space–frequency analysis of signals” [7–9]. The successful use of the Wigner distribution in the theory of signals was due to its “good” mathematical characteristics, especially by its representative characteristics that are basic in the renewal of signal intensity distribution.

Within the framework of the given investigation, of a variety of space–frequency representations we single out two basic distributions by Wigner and Weyl, which are widely used in the theory of signals for solving diverse physical problems [5,10]. The investigation of the signal characteristics is carried out on the basis of comparison with its displaced analogues.

For present day, various types of space–frequency distributions are successfully used for the analysis of nonstationary signals [4,5]. Many of such distributions are characterized by advantages as well as by disadvantages in use in various fields of physics. The Wigner and Weyl distributions are widely used in space–frequency analysis and, in particular, in optical information processing systems. It is well-known that the distributions possess such characteristics that are successfully used for description of many optical systems. The spectrum of applications of these distributions is extremely wide. They are used, in particular, in the theory of optical lens systems, theory of communication, hydrolocation and other fields [6,10,11]. The research of last years

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proved the efficiency of use of space–frequency distributions in biology and medicine; especially the Wigner distribution was successfully used for renewal of volumetric structure of objects within the framework of optical tomography [12–14]. One of the promising investigation directions within the space–frequency processing of signals is study of the properties of novel space–frequency distributions, with the aim of their further applications in different areas of physics and medicine. Unfortunately, some space–frequency distributions often do not meet demands raised by one or another specific application. In this relation, many of the existing distributions need generalization or improvements when applied to a given problem. During the second half of the past century and the beginning of this one, a clear tendency has been observed towards generalization of different space–frequency distributions. The first attempt of such a generalization was due to Cohen [15] as long ago as in 1966. The author introduced a number of quasiprobability distributions that provided proper quantum mechanical marginal distributions. Within the limits of this research, the Wigner distribution was examined as a special case. The next step was done by De Bruijn in 1973 [16]. His work was devoted to elaboration of the theory of generalized functions, with application concerned Wigner and Weyl distributions. Summarizing the results of numerous investigations, Cohen [4,6] suggested a generalized distribution involving a certain kernel:

$$C(x, \omega, \Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\left(y + \frac{x_0}{2}\right) f^*\left(y - \frac{x_0}{2}\right) \times \Phi(x, \omega) e^{-i(\omega x_0 - \omega'_0 x + \omega'_0 y)} dy dx_0 d\omega'_0. \quad (1)$$

Depending on the form of the kernel $\Phi(x, \omega)$, this distribution degenerates into one of the known distributions (Wigner, Weyl, Woodward, Kirkwood, Page, Mark, etc.).

In general, expression (1) describes a class of space–frequency distributions later named Cohen's class. This class contains known distributions as well as a set of still unknown distributions, which also satisfy all necessary requirements of existing distributions. The theory of generalization of space–frequency distributions was developed by Mertins, who also is a famous specialist in the theory of signals. In his monograph "Signal Analysis" [10] he singled out this topic into a separate section "General space–frequency distributions". The author stated that the Wigner distribution serves as an excellent tool for space–frequency analysis as long as a linear dependence is kept between the instantaneous coordinates and frequencies. Otherwise, a need in generalization appears, general principles of which are described in detail in the mentioned work.

Among numerous recent studies related to generalizing space–frequency distributions, we should mention only the most typical ones. The Ph.D. thesis by Durak "Novel time–frequency analysis technique for deterministic signals" [17] is one of such studies, where a close attention was paid to generalized distributions and their additional parameters were introduced.

In the present work we try to use relationships between the Wigner and Weyl distributions with the aim of "mixing" them into a single, more general distribution. Up to date, it has been revealed that the two distributions are related by a double Fourier transform. The results obtained by us allow tracing transformation of one of the distributions into the other, while changing the distribution parameter t . This generalized distribution generates a whole set of new distributions formed in the process of switching between the basic distributions. The latter fact may be important from the viewpoint of possible practical applications. For the present day a choice between the Weyl and Wigner distributions remains ambiguous. Each of them has its own scheme for reconstruction of signal intensity distribution. The scheme adopted for the Wigner distribution includes calculating the

marginal distributions [4]. The Weyl distribution provides much simpler reconstruction scheme, owing to simpler mathematical transformations [18,19]. Traditionally, the Wigner distribution has been used in a large majority of studies performed within the field. Introduction of the mutual distribution would mean a possibility for calculating 'mixed' states and determining the necessary contributions of each of the limiting distributions. As stated above, there appears a possibility for generalization of distributions concerning various applied problems. However, only Chountasis suggested an approach [20] that enables transitions between the Wigner and Weyl distributions. Such distributions play an important role in the analysis of phase space and, moreover, can be immediately applied in the Wigner tomography [12,14]. Chountasis and co-authors [20–22] developed a general distribution based on the Wigner formalism, which involves an additional parameter θ . This study was performed in frame of quantum-mechanical formalism.

The problem of calculation of a classical analogue of this generalized distribution remains urgent. It may be constructed based on the results [20] or using the formalism of Weyl distribution, as was done by the present author when studying the properties of fractional Fourier transform [23]. Similar to the works [18,19], the author has employed peculiarities of reconstruction of signal intensity based upon the Weyl distribution. Meanwhile, only this reconstruction scheme is realized experimentally in the real optical schemes [23].

2. Transformations between space–frequency distributions

For today, there are many different types of space–frequency (space–time) distributions that are both well known and widely exploited. Most spread of them are the Wigner and the Weyl distributions.

Consider two signals described by functions $f_1(x)$ and $f_2(x)$, which, in turn, are assumed to possess Fourier transforms $F_1(\omega)$ and $F_2(\omega)$, respectively. The Wigner distribution of such signals has the form

$$W_{f_1 f_2^*}(x, \omega) = \int_{-\infty}^{+\infty} f_1\left(x + \frac{x_0}{2}\right) f_2^*\left(x - \frac{x_0}{2}\right) e^{-i\omega x_0} dx_0, \quad (2)$$

where $f_2^*(y)$ is the complex conjugate of $f(y)$. The expression

$$F_{f_1 f_2^*}(x, x_0) = f_1\left(x + \frac{x_0}{2}\right) f_2^*\left(x - \frac{x_0}{2}\right) \quad (3)$$

is referred to as the autocorrelation function and can also be regarded as a kind of distribution. The Weyl distribution is written as

$$A_{f_1 f_2^*}(x_0, \omega_0) = \int_{-\infty}^{+\infty} f_1\left(x + \frac{x_0}{2}\right) f_2^*\left(x - \frac{x_0}{2}\right) e^{-ix\omega_0} dx. \quad (4)$$

The quantities x and ω , on which the Wigner distribution is dependent, are average values of space coordinate and frequency, respectively. Another pair of variables, x_0 and ω_0 , describes deviations from the average values. The authors of work [24] consider these two pairs of variables as conceptually different. Variables x and ω are called "slow", while x_0 and ω_0 are called "rapid deviations".

By definition, the Fourier transform for $f(x)$ is

$$F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx; \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} d\omega.$$

Therefore, we can introduce corresponding autocorrelation function:

$$\Phi_{f_1 f_2^*}(\omega, \omega_0) = F_1\left(\omega - \frac{\omega_0}{2}\right) F_2^*\left(\omega + \frac{\omega_0}{2}\right) \quad (5)$$

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