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Analytical localized wave solutions of the generalized nonautonomous nonlinear Schrödinger equation with Gaussian shaped nonlinearity

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ABSTRACT

We construct analytical localized wave solutions to the generalized nonautonomous nonlinear Schrödinger equation with Gaussian shaped nonlinearity and trapping potentials by using a similarity transformation technique. Our results show that analytical localized wave solutions possess $n-1$ zeros where their existence requires some restrictive conditions corresponding to the dispersion coefficient, the Gaussian shaped nonlinearity, the gain (loss) coefficient, and the trapping potential. In addition, the stability analysis of the solutions is discussed numerically.

1. Introduction

The nonlinear Schrödinger equation (NLSE) is one of the most important nonlinear models that emerges in many physical phenomena, including nonlinear optics [1], Bose–Einstein condensates (BECs) [2], plasma physics [3], hydrodynamics [4], and some organic materials [5]. Various types of solutions to NLSE are found of great interest due to their applications in physical systems, such as bright (dark) solitons [6], periodic traveling waves [7], and localized waves [8].

In recent years, the nonautonomous system, a system received some form of external time-dependent or space-dependent force, has attracted extensive attention due to its interesting features and potential applications [9,10]. Solutions to such model are often called nonautonomous waves or nonautonomous solitons. One of the representative examples is the nonautonomous NLSE (or the NLSE with varied coefficients). Several methods are applied to solving this model to obtain numerical and analytical solutions [11–13]. Among these methods, the similarity transformation technique, in which transformation parameters can be determined by a set of differential equations, has been applied successfully to the nonautonomous NLSE [8]. In addition, it helps to produce selected solutions in analytical form, which may be important for a variety of applications.

Our interest is focused on a generalized nonautonomous NLSE with an external potential describing soliton management in nonlinear optics [14]. This model in the one-dimensional case can be given by the following dimensionless form for complex function $\psi(x,t)$:

$$i\frac{\partial\psi}{\partial t}+f(x,t)\frac{\partial^2\psi}{\partial x^2}+g(x,t)|\psi|^2\psi+V(x,t)\psi+i\gamma(x,t)\psi=0. \tag{1}$$

Here x is the transverse variable, t is the longitudinal variable. The functions $f(x,t)$,$g(x,t)$,$\gamma(x,t)$ are, respectively, the dispersion coefficient, the nonlinearity, and the gain (loss) coefficient, and $V(x,t)$ is the external potential function. The boundary condition requires $\psi(x\to\pm\infty)=0$. Usually, the dispersion coefficient $f(x,t)$ is related to the linear refractive index n_0 that is usually nonuniform distributed in the longitudinal direction (propagation direction) in nonlinear media. It is expected that, our model (1), will be existed in the nonlinear media whose transverse direction is also inhomogeneous, i.e., the linear refractive index $n_0=n_0(x,t)$. In the context of BECs, Eq. (1) describes the dynamics of matter-wave solitons [15] whose management can be realized by adjusting the related control parameters via the technique of Feshbach resonance [16,17]. t and x, in this case, represent the time and spatial coordinate, respectively.

Recently the generalized nonautonomous NLSE (1) has been extensively investigated in the literature [18–21] and some useful techniques have been explored. Specifically, Eq. (1) in the case of $\gamma=0$ was treated in Ref. [18] by using the Painlevé analysis and the symmetry reduction and a classes of exact solutions were found. In Refs. [19,20] we have studied the generalized nonautonomous (cubic-quintic) NLSE with time- and space-dependent distributed coefficients and external potentials and given the analytical bright multisoliton (solitary wave) solutions to it.

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Recently, a work in Ref. [21] also presented the integrability conditions of Eq. (1) by employing the Lax pair and similarity transformation methods.

In this paper, we consider that the nonlinearity coefficient in Eq. (1) is Gaussian shaped distribution, which is important in the case of BECs with controlled optical interactions [22] and may be exist in the nonlinear media when its transverse and longitudinal directions are nonuniform distribution. The trapping potential induced by the Gaussian shaped nonlinearity is then presented by using a similarity transformation technique. Thus, such trapping potential may be designed in the nonlinear media whose transverse and longitudinal directions are nonuniform distribution and the nonlinearity coefficient is Gaussian shaped distribution.

We start the reduction of Eq. (1) in the next section, where a trapping potential supported by the Gaussian shaped nonlinearity is presented, following as close as possible the recent works [8]. In Section 3, an infinite number of exact localized wave solutions induced by the Gaussian shaped nonlinearity and the trapping potential are obtained. We also discuss their physical applications and predict their possibility existences in nonlinear systems. In Section 4, stability analysis of the solutions is discussed numerically. We finish the work in Section 5, where we make the conclusions.

2. Similarity reduction

We suppose the Gaussian shaped nonlinearity as $g(x,t)=\alpha e^{-\xi^2/b^2}$, where $\xi(x,t)=\alpha(t)x$ with $\alpha(t)$ being an positive definite function of time. Our goal is to reduce Eq. (1) to the stationary NLSE

$$E\Phi=-\Phi_{XX}+G|\Phi|^2\Phi, \tag{2}$$

where $\Phi(X)$ is a real function to be determined, E is the eigenvalue of the nonlinear equation, and G is a constant.

To connect solutions of Eq. (1) with those of Eq. (2) we can assume the wave function as

$$\psi(x,t)=\rho(x,t)\Phi(X(x,t))e^{i\phi(x,t)}, \tag{3}$$

where $X(x,t)=\int_{-\infty}^{\xi}e^{-\xi'^2/b^2}\,d\xi'$. The substitution of Eq. (3) into Eq. (1) then leads a system of equations when requiring $\Phi(X)$ to satisfy Eq. (2). Solving these equations, one finds the following solutions:

$$\rho(x,t)=\sqrt{\frac{a}{\alpha}}e^{\xi^2/b^2}, \tag{4a}$$

$$\phi(x,t)=\frac{1}{2}Ga^{-1}\alpha\alpha_t\int_0^x e^{-2\xi^2/b^2}x'\,dx', \tag{4b}$$

$$f(x,t)=-\frac{a}{G\alpha^2}e^{\xi^2/b^2}, \tag{4c}$$

$$\gamma(x,t)=\frac{\alpha_t}{\alpha}-\frac{a_t}{2a}-\frac{2\alpha\alpha_t}{b^2}x^2, \tag{4d}$$

where $a=a(t)$ is arbitrary function of time. Note that the integral function of time in Eq. (4b) is chosen to be zero. Now the trapping potential supported by the Gaussian shaped nonlinearity is given by (for simplicity, we take $a=1$, which corresponds to the case in Section 3.1)

$$V(x,t)=l(t)+h_1(\xi,t)e^{2\xi^2/b^2}+h_2(\xi,t)e^{-2\xi^2/b^2}, \tag{5}$$

with

$$l(t)=\frac{Gb^2}{8\alpha^2}(\alpha\alpha_{tt}-\alpha_t^2)-\frac{E}{G},$$

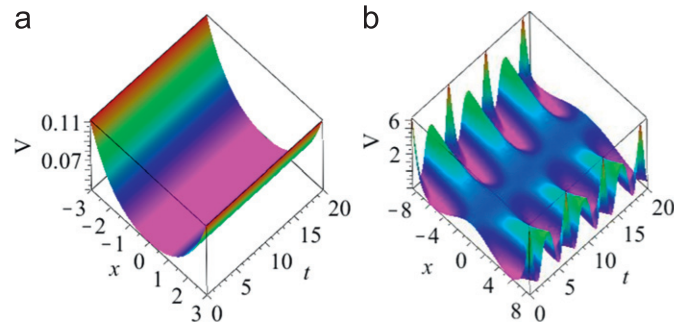

Fig. 1. Plots of the trapping potential supported by the Gaussian shaped nonlinearity with (a) $\epsilon=0$ and (b) $\epsilon=0.5$. Other parameters are $\omega_0=G=1$, $E=0, b=8$.

$$h_1(\xi,t)=\frac{1+\xi^2/b^2}{Gb^2},$$

$$h_2(\xi,t)=\frac{Gb^2[\alpha_t^2(1+2\xi^2/b^2)-\alpha\alpha_{tt}]}{8\alpha^2}.$$

This is a trapping potential in the form of a combination of time-modulation in trigonometric form ($l(t)$ in Eq. (5)) and Gaussian terms varying in space and time (the latter two terms on the right hand in Eq. (5)), as plotted in Fig. 1. When $\epsilon=0$, its profile is displayed in Fig. 1(a); when $\epsilon\neq 0$, its profile is displayed in Fig. 1(b), which shows a breathing behavior in the vicinity of $x=0$. It is seen that the trapping potential is not arbitrary, but related to $g(x,t), f(x,t)$ and $\gamma(x,t)$ via the parameters $a(t), \alpha(t)$. Thus, Eqs. (4) and (5) should be understood as integrability conditions on Eq. (1) for exact solutions by the present method.

When $G<0$, corresponding to the attractive nonlinearity, an exact solution to Eq. (2) is chosen as

$$\Phi(X)=\sqrt{(E-\lambda^2)/G}\,\mathrm{cn}(\lambda X-X_0,\sqrt{(\lambda^2-E)/2\lambda^2}), \tag{6}$$

where λ and X_0 are arbitrary constants, and cn and sn (below) are the Jacobian elliptic functions. The restriction on $\Phi(X)$ requires $-\lambda^2\leq E<\lambda^2$.

When $G>0$, corresponding to the repulsive nonlinearity, a nontrivial exact solution to Eq. (2) is

$$\Phi(X)=\sqrt{2(E-\lambda^2)/G}\,\mathrm{sn}(\lambda X-X_0,\sqrt{E/\lambda^2-1}), \tag{7}$$

where $\lambda^2<E<2\lambda^2$.

Next, to investigate analytical solutions we choose

$$\alpha(t)=1+\epsilon\cos(\omega_0 t), \tag{8}$$

where $\epsilon\in(-1,1)$, $\omega_0\in\mathbb{R}$.

3. Analytical localized wave solutions

In this section, we investigate the dynamics of the analytical localized nonlinear wave solutions (6)–(7) of the generalized nonautonomous NLSE (1). We also discuss their physical applications and predict their possibility existences in nonlinear systems. It is interesting that the localized wave solutions under the Gaussian shaped nonlinearity present different features for the choice of a [α is the trigonometric form mentioned in Eq. (8)]. We mainly focus on the following two cases:

3.1. Case of $a=1$

In this case, We have $\rho(x,t)=\sqrt{\alpha^{-1}e^{\xi^2/b^2}}$. To obtain localized wave solution of Eq. (1), we can impose $\Phi(X(\pm\infty))=0$ to satisfy the boundary condition $\psi(\pm\infty)=0$. Obviously, the condition