Optics Communications 285 (2012) 510–516

Contents lists available at SciVerse ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Study of Gaussian and Bessel beam propagation using a new analytic approach

C.A. Dartora [a,*], K.Z. Nobrega [b]

[a] Electrical Engineering Department, Federal University of Parana (UFPR), Brazil
[b] Electrical Engineering Department, Federal Institute of Technological Education of Maranhao (IFMA), Brazil

ARTICLE INFO

Article history:
Received 4 June 2011
Received in revised form 3 November 2011
Accepted 7 November 2011
Available online 21 November 2011

Keywords:
Paraxial wave equation
BESSEL beams
Fourier-Bessel series

ABSTRACT

The main feature of Bessel beams realized in practice is their ability to resist diffractive effects over distances exceeding the usual diffraction length. The theory and experimental demonstration of such waves can be traced back to the seminal work of Durnin and co-workers already in 1987.

Despite that fact, to the best of our knowledge, the study of propagation of apertured Bessel beams found no solution in closed analytic form and it often leads to the numerical evaluation of diffraction integrals, which can be very awkward. In the context of paraxial optics, wave propagation in lossless media is described by an equation similar to the non-relativistic Schrödinger equation of quantum mechanics, but replacing the time t in quantum mechanics by the longitudinal coordinate z. Thus, the same mathematical methods can be employed in both cases. Using Bessel functions of the first kind as basis functions in a Hilbert space, here we present a new approach where it is possible to expand the optical wave field in a series, allowing to obtain analytic expressions for the propagation of any given initial field distribution. To demonstrate the robustness of the method two cases were taken into account: Gaussian and zeroth-order Bessel beam propagation.

1. Introduction

The main feature of pseudo non-diffracting Bessel beams is their ability to resist diffractive effects over distances exceeding the usual diffraction length. The theory and experimental demonstration of such waves can be traced back to the seminal work of Durnin and co-workers already in 1987 [1], opening a new era in the study of diffraction phenomena. Since then, the interest in studying non-diffracting beams has not ceased to grow, due to their potential applications in fields of metrology, tweezing, laser surgery and so on. Nowadays, the study of pseudo non-diffracting beams reached its maturity and numerous examples of such beams have been both mathematically and experimentally demonstrated in optics [2–17]. Despite that fact, to the best of our knowledge, the study of propagation of apertured Bessel beams found only a few solutions in closed analytic form [18–20] and it often leads to the numerical evaluation of diffraction integrals, which can be very awkward. Thus, a convergent series allowing the analysis of apertured Bessel beam propagation in free space in closed analytic form is highly desirable.

It is well know that Optics and Quantum Mechanics have a lot in common. From historical point of view, it was an old formulation based on Optics by Hamilton and Jacobi that inspired Erwin Schrödinger to put forward his wave mechanical version of quantum mechanics [21]. In that occasion, Hamilton and Jacobi were searching for an identification of a particle's trajectory with the gradient of constant phase surfaces $S(x,t)$ of a wave $\psi(x,t)=\varphi(x)e^{-i[\omega t-S(x,t)]}$.

Indeed, at optical frequencies in the so-called paraxial regime the propagation of electromagnetic fields is described by a wave equation analogous to the non-relativistic Schrödinger equation of quantum mechanics, with the role of time coordinate t in quantum mechanics played by the longitudinal coordinate z in the paraxial equation. Thus, the usual mathematical methods of quantum mechanics can be promptly applied to the study of paraxial optics [22,23]. For instance, group theoretical methods [24], commonly used to describe the band structure of a solid in condensed matter physics, have been also applied to the study of photonic band structure in the area of photonic crystals [25–29].

In the present contribution our main goal is to obtain an analytic solution in closed form for the study of apertured Bessel beams in free space. To do that we benefit from the above mentioned analogies between paraxial optics and quantum mechanics, taking into account the properties of completeness and closure of basis functions in a Hilbert space, expanding the wave function in a convenient basis, which allows one to obtain analytic expressions for the propagation of any given initial field distribution. In cylindric coordinates the basis is composed of Bessel functions of the first kind, leading to a Bessel–Fourier series. In order to show the stability and convergence of our method, we propagate a Gaussian and a truncated zeroth-order Bessel beam, whose behavior is extensively related in the current literature.

The content of this article can be described as follows: in the next section we will put forward the theoretical framework, pointing out

* Corresponding author.
E-mail address: cadartora@yahoo.com.br (C.A. Dartora).

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doi:10.1016/j.optcom.2011.11.015

the equivalence between paraxial optics and quantum mechanics, expanding any initial field distribution in terms of a convenient complete basis of functions in a Hilbert space. A Gaussian beam will be propagated and the obtained results are compared to known analytical solutions. In Section 3, we will study the propagation of an apertured zeroth-order Bessel beam in free space, using the Bessel functions as the basis for a Fourier–Bessel series expansion of the propagated field and discuss the convergence and properties of the series. To finish, in the last section, a few conclusions and remarks are added.

2. Theoretical framework

Let us start this section through a brief review of main aspects of non-relativistic Schrödinger equation, shown below [30]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \ , \tag{1}$$

being ψ as the quantum-mechanical complex wavefunction, $\hbar = h/(2\pi)$ the Planck's constant, m as the particle mass and V as the potential energy. Defining the Hamiltonian operator as $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$, one has an eigenvalue problem of the form $\hat{H}\psi_n = E_n\psi_n$ for functions of the form $\psi_n(x,y,z,t) = e^{-iE_nt/\hbar}\psi_n(x,y,z,0)$, given the boundary conditions $\psi(\infty)\to 0$ and $\int\psi^\dagger\psi d^3\mathbf{x} = 1$, leading to the knowledge of an orthonormal set of basis functions in a complex vector space known as a Hilbert space. Such basis allows to expand any quantum mechanical field ψ in the following way:

$$\psi(x,y,z,t) = \sum_n c_n(0)e^{-iE_nt/\hbar}\psi_n(x,y,z,0),$$

being c_n as complex amplitudes and, in principle $\int\psi_m^\dagger\psi_n d^3\mathbf{x} = \delta_{mn}$, where δ_{mn} is the Kronecker delta function. It is important to notice that $\sum_n|c_n|^2 = 1$, which means that the norm of the abstract vector ψ is preserved.

After those considerations on quantum mechanics, let's turn the attention to the wave equation in frequency domain, the so-called Helmholtz Equation:

$$\left(\nabla^2 + k^2\right)\Phi(x,y,z) = 0 \ , \tag{2}$$

where $k^2 = n^2\omega^2/c^2 = 2\pi/\lambda$, $n(x,y,z,\omega)$ is the refractive index of the medium and the wave function Φ is used to merge the time harmonic electric and magnetic fields, **E** and **H**, respectively, into a single entity, as follows:

$$\Phi = \begin{pmatrix}\mathbf{E}\\ Z\mathbf{H}\end{pmatrix}e^{-i\omega t} \ , \tag{3}$$

being $Z = \sqrt{\mu/\varepsilon}$ as the characteristic impedance of a medium with magnetic permeability μ and dielectric permissivity ε. Next, we write $\Phi(x,y,z) = \Psi(x,y,z)e^{i\beta z}$, removing the rapid variations of the wave function along the z-axis, which we assume to be the longitudinal coordinate. Neglecting second order derivatives of $\Psi(x,y,z)$ with respect to z (paraxial regime) we get:

$$i\frac{\partial\Psi}{\partial z} = -\frac{1}{2\beta}\left[\nabla_\perp^2\Psi + \left(k^2 - \beta^2\right)\Psi\right], \tag{4}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse laplacian operator and β is the propagation constant along the longitudinal axis. The above expression is the so-called paraxial wave equation.

Looking at Eqs. (1) and (4), one can notice enormous similarities between them, with the role of time coordinate t in Eq. (1) being played by the longitudinal coordinate z in Eq. (4), the full laplacian operator ∇^2 in the quantum mechanical Schrödinger equation is replaced by its transverse version $\nabla_\perp^2$ in the paraxial wave equation and the potential energy $V(x,y,z,t)$ corresponding to $[k^2(x,y,z) - \beta^2]/(2\beta)$. In this way, the same mathematical methods employed in quantum mechanics can be used in the study of paraxial optics.

In free space we can make $k = \beta$ and the paraxial wave Eq. (4) reduces to:

$$i\frac{\partial\Psi}{\partial z} = -\frac{1}{2k}\nabla_\perp^2\Psi \ . \tag{5}$$

Obviously the so-called ideal or non-diffracting Bessel beams, defined as $\Psi(\rho,\phi,0) = A_m(k_\rho)J_m(k_\rho\rho)e^{im\phi}$, being $m = 0,1,2,3\ldots$ an integer and $0 \le k_\rho < \infty$ a real number, do form a complete set of basis functions for the solution of the wave equation in cylindric coordinates (ρ,ϕ,z), as is the case of uniform plane waves in cartesian coordinates. A formal integral solution of the above equation in cylindric coordinates, using the Bessel beams in unlimited free space ($0 \le \rho \le \infty$ and $0 \le \phi \le 2\pi$) as the basis is given by:

$$\Psi(\rho,\phi,z) = \sum_{m=0}^{\infty} e^{im\phi}\int_0^\infty k_\rho dk_\rho e^{-i\frac{k_\rho^2}{2k}z}A_m\left(k_\rho\right)J_m\left(k_\rho\rho\right), \tag{6}$$

where the coefficients are determined in the following way:

$$A_m\left(k_\rho\right) = \frac{1}{(2\pi)^2}\int_0^\infty\rho d\rho\int_0^{2\pi}d\phi\Psi(\rho,\phi,0)J_m\left(k_\rho\rho\right)e^{-im\phi} \ , \tag{7}$$

being $\Psi(\rho,\phi,0)$ as the initial field distribution. An ideal Bessel beam of order n is obtained by making $A_m(\alpha) = 2\pi\delta_{mn}\delta(k_\rho - \alpha)$, being δ_{mn} the Kronecker delta function and $\delta(k_\rho - \alpha)$ the Dirac delta function. In practice such ideal Bessel beams extending over the range $0 \le \rho \le \infty$ cannot be realized experimentally because they demand an infinite amount of energy to be produced. By contrast, finite aperture realizations of Bessel beams are feasible, resisting the diffractive effects over propagated distances greater than the Gaussian beams, for example. The distance over which they can be considered invariant exceeds the usual diffraction length [1]. However, for the truncated Bessel beam ($\Psi(\rho,\phi,0) = J_m(\alpha\rho)e^{im\phi}, \rho \le a$ and $\Psi = 0, \rho > a$) corresponding to the real world, it turns out that expressions (6) and (7) cannot be solve altogether analytically.

Our primary concern here is to obtain analytical expressions for the propagation of a truncated Bessel beam, which cannot be obtained using a complete basis in the physical space $0 \le \rho \le \infty$. But since the practical apertured Bessel beam is truncated, existing initially for $0 \le \rho \le a$ only, we will be able to obtain a series in closed analytic form by weakening the general requirement of a well behaved solution valid in whole physical domain $0 \le \rho \le \infty$, limiting the region of interest to the physical region $0 \le \rho \le \rho_0$, being $\rho_0 > a$, resulting in a discrete set of basis functions subjected to a Dirichlet boundary condition at $\rho = \rho_0$, instead of a continuous set of basis functions. Following, let's show how it is possible to do that.

Defining the differential operator $\hat{H}_0 = -\frac{1}{2k}\nabla_\perp^2$, which is the analog of the Hamiltonian in quantum mechanics, being hermitian and thus preserves the norm of the wave function ψ, we can write the formal solution to Eq. (5) as follows:

$$\Psi(\rho,\phi,z) = e^{-i\hat{H}_0 z}\Psi(\rho,\phi,0). \tag{8}$$

Given the initial field $\Psi(\rho,\phi,0)$, a direct calculation can be done by expanding the exponential in Taylor series, but such a procedure usually leads to slowly converging solutions. In order to overcome such a difficulty we will use the eigenfunctions of $\hat{H}_0$ in the domain $0 \le \rho \le \rho_0$, $0 \le \phi \le 2\pi$ subjected to a Dirichlet boundary condition at $\rho = \rho_0$, forming a complete vector space of functions to expand the initial field. In cylindric coordinates (ρ,ϕ), the eigenvalue problem