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Exact self-similar wave solutions for the generalized $(3+1)$-dimensional cubic–quintic nonlinear Schröinger equation with distributed coefficients

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ABSTRACT

In this paper, a similarity transformation is presented to reduce the generalized $(3+1)$-dimensional cubic–quintic nonlinear Schrödinger equation with distributed coefficients to the related constant-coefficients one. Then a number of spatiotemporal self-similar wave solutions are constructed. Under the specific choice of the dispersion, cubic and quintic nonlinearities, phase modulation and the gain/loss, we investigate the dynamical behaviors of those spatiotemporal self-similar waves in an inhomogeneous optical fiber media.

1. Introduction

In the past decade, there has been a great deal of theoretical and experimental investigations in models based on nonlinear Schrödinger equation (NLSE), which can be applied widely to many branches of physics and applied mathematics, including nonlinear quantum field theory, condensed matter and plasma physics, nonlinear optics and quantum electronics, and so on. Due to the potential application of optical solitons in long distance telecommunication, more and more attention has been attracted to different versions of NLSE which can describe nonlinear management and dispersion management for temporal or spatial optical solitons, soliton lasers, and ultrafast soliton switches in nonlinear fibers [1,2]. Many methods have been developed to obtain the soliton solutions of various NLSEs with various nonlinearities and dispersions, and the dynamics of the excitation of the solitons was discussed [3,4,5,6,7,8,9,10,11,12]. For example, Serkin et al. presented a system method to construct analytical solutions for $(1+)$-dimensional variable-coefficients NLSE and discussed nonautonomous soliton dispersion management [3]; Dai et al. investigated the similariton transmission control in the dispersion decreasing fiber [4]; Li et al. investigated interactions of dark solitons in photovoltaic photorefractive crystals with diffusion nonlinearity [5]. Recently, a kind of NLSE called Gross–Pitaevskii (GP) equation is presented and used to describe a dilute gas of weakly interacting atomic particles. The GP equation has become an important theoretical tool in recent Bose–Einstein condensates (BECs) experiments [13,14], the various nonlinear excitations of matter-wave solitons have been observed and studied [15,16,17,18].

The above studies have stimulated a large amount of research activities on nonlinear optics and BECs. Especially, great interest is focused on the $(2+1)$dimensional (D) and $(3+1)$D NLSEs from various view points [19,20,21,22,23,24,25], which provide excellent proving grounds for exploring higher-dimensional nonlinear systems with distributed coefficients. In this paper, we will focus our interests on a generalized $(3+1)$-dimensional (D) cubic–quintic nonlinear Schröinger equation (CQNLSE) with distributed coefficients in an inhomogeneous optical fiber media. Historically, this model was first studied by Serkin et al., where the topological quasi-soliton solutions for the inhomogeneous CQNLSE were found [26]. Recently, various solitary-wave solutions and modulational instability are studied in [27,28,29,30]. Furthermore, nonlinear optical organic materials and waveguides have been investigated as the key elements for future telecommunication and photonic technologies, where thin films of polydiacetylene para-toluene sulfonate exhibit the cubic–quintic nonlinearity [31].

The paper is organized as follows. In Section 2, by making use of the symmetry group direct method [32,33] and symbolic computation, we present a similarity transformation to the generalized $(3+1)$D CQNLSE, where it can be reduced to the related constant-coefficient $(3+1)$D CQNLSE. In Section 3, some exact spatiotemporal self-similar solutions of the generalized $(3+1)$D CQNLSE are constructed through the similarity transformation and the well-known Jacobi elliptic function solutions of the related constant-coefficient $(3+1)$D CQNLSE. In Section 4, we investigate the dynamics of self-similar waves in dispersion decreasing fiber and dispersion changing periodically fiber. Finally, a summary is given in Section 5.

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2. Similarity transformation of (3+1)D CQNLSE with variable coefficients

The evolution of a slowly varying wave packet envelope $u(z,x,y,t)$ in a diffractive nonlinear Kerr medium with anomalous dispersion, in the paraxial approximation, can be written as

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{\rho(z)}{2}\nabla^2 u+g_3(z)\left|u\right|^2 u+g_5(z)\left|u\right|^4 u+M(z)r^2 u-\mathrm{i}\gamma(z)u=0, \tag{1}$$

where $\nabla^2=\partial_{xx}+\partial_{yy}+\partial_{tt}$ and $r^2=x^2+y^2+z^2$, z is the propagation coordinate, and t the reduced time, i.e., time in the frame of reference moving with the wave packet. All coordinates are made dimensionless by the choice of coefficients. $\rho(z)$ is the group velocity diffraction or dispersion parameter, and $g_3(z)$ and $g_5(z)$ represent cubic and quintic nonlinearities, respectively. The functions $M(z)$ and $\gamma(z)$ are related to phase modulation, and gain/loss coefficients, respectively. The generalized CQNLSE is of considerable importance. When $g_5(z)=M(z)=0$, it describes the full spatiotemporal optical solitons, or light bullets, in (3+1)D [22,23]. When $M(z)=0$, Jiang et al. investigated the spatiotemporal self-similar solutions and discussed the related dynamical behaviors [24]. When $\rho(z)=\gamma(z)=0$, Gao et al. derived some exact solutions by a simple similarity transformation [25].

On the basis of the idea of the symmetry group direct method, we can assume the solutions of Eq. (1) is in the form of

$$u=\delta(z)\psi(\zeta,\xi,\eta,\tau)exp(i\phi), \tag{2}$$

where ζ,ξ,η,τ are all real functions of $\{z,x,y,t\}$, $\delta(z)$ and $\phi=\varphi(z,x,y,t)$ are real functions and the complex function ψ satisfies the (3+1)-dimensional constant-coefficient CQNLSE

$$\mathrm{i}\frac{\partial\psi}{\partial\zeta}+k_1\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}+\frac{\partial^2\psi}{\partial\tau^2}\right)+k_3\left|\psi\right|^2\psi+k_5\left|\psi\right|^4\psi=0, \tag{3}$$

where k_1,k_3,k_5 are real constants. Here, it is necessary to point out that according to Lie group theory, we should assume ψ to be satisfied the same equation as Eq. (1) with independent variables $\{\zeta,\xi,\eta,\tau\}$ and variable-coefficients $\{\rho(\zeta),g_3(\zeta),g_5(\zeta),M(\zeta),\gamma(\zeta)\}$. For questions discussed in this paper, we set ψ to be satisfied with the simpler form (Eq. (3)).

Substituting Eq. (2) into Eq. (1), and eliminating ψ_ζ by Eq. (3), we can obtain a polynomial differential equations with respect to ψ and its derivatives. Then collecting their coefficients of ψ and its derivatives and separating the real part and imaginary part, we can obtain a set of nonlinear PDEs. An equation among them is $\zeta_x^2+\zeta_y^2+\zeta_t^2=0$, so we can obtain

$$\zeta=\theta(z). \tag{4}$$

Then substituting Eq. (4) into the set of nonlinear PDEs, we can obtain the following set of PDEs:

$$\begin{aligned}
&\rho\left(n_x^2+n_y^2+n_z^2\right)-k_1\dot{\theta}=0,\ (n=\xi,\eta,\tau),\\
&\rho\left[2\left(n_x\varphi_x+n_y\varphi_y+n_t\varphi_t\right)-i\left(n_{xx}+n_{yy}+n_{tt}\right)\right]+2n_z=0,\ (n=\xi,\eta,\tau),\\
&\rho\delta\left(\varphi_{xx}+\varphi_{yy}+\varphi_{tt}\right)+2\left(\dot{\delta}-\delta\gamma\right)=0,\\
&\rho\delta\left(\varphi_{xx}+\varphi_{yy}+\varphi_{tt}\right)+2\delta\left[\varphi_z-M\left(x^2+y^2+z^2\right)\right]=0,\\
&\xi_x\eta_x+\xi_y\eta_y+\xi_t\eta_t=0,\\
&\eta_x\tau_x+\eta_y\tau_y+\eta_t\tau_t=0,\\
&\tau_x\xi_x+\tau_y\xi_y+\tau_t\xi_t=0,\\
&k_j\dot{\theta}-g_j\delta^{j-1}=0,\ (j=3,5)
\end{aligned} \tag{5}$$

where the subscript means its derivative with respect to x,y,t,z, and the dot over the function means its derivative with respect to time z. With the help of symbolic computation, we obtain the following solutions of Eqs. (5)

$$\begin{aligned}
&\delta=\delta_0\alpha^{3/2}exp\left(\int_0^z\gamma\mathrm{d}z\right),\theta=\frac{s}{k_1}\int_0^z\rho\alpha^2\mathrm{d}z,\\
&\xi=\alpha(a_1x+a_2y+a_3t)-\sum_{j=1}^3 a_je_j\int_0^z\rho\alpha^2\mathrm{d}z,\\
&\eta=\alpha(b_1x+b_2y+b_3t)-\sum_{j=1}^3 b_je_j\int_0^z\rho\alpha^2\mathrm{d}z,\\
&\tau=\alpha(c_1x+c_2y+c_3t)-\sum_{j=1}^3 c_je_j\int_0^z\rho\alpha^2\mathrm{d}z,\\
&\phi=-\frac{\dot{\alpha}\left(x^2+y^2+t^2\right)}{2\rho\alpha}+\alpha(e_1x+e_2y+e_3t)\\
&\qquad-\frac{1}{2}\left(e_1^2+e_2^2+e_3^2\right)\int_0^z\rho\alpha^2\mathrm{d}z,
\end{aligned} \tag{6}$$

where $\{\rho(z),M(z),\alpha=\alpha(z),g_3(z),g_5(z)\}$ satisfy the following conditions:

$$2M\rho^2\alpha^2+\rho\alpha\ddot{\alpha}-2\rho\dot{\alpha}^2-\alpha\dot{\alpha}\dot{\rho}=0,g_3=\frac{k_3\dot{\theta}}{\delta^2},g_5(z)=\frac{k_5\dot{\theta}}{\delta^4}, \tag{7}$$

and $\{a_i, b_i, c_i\ (i=1,2,3)\}$ satisfy the following algebraic equations

$$\sum_{j=1}^3 a_j^2=\sum_{j=1}^3 b_j^2=\sum_{j=1}^3 c_j^2=s,\sum_{j=1}^3 a_jb_j=\sum_{j=1}^3 b_jc_j=\sum_{j=1}^3 c_ja_j=0, \tag{8}$$

and $\delta_0,e_i(i=1,2,3)$ are arbitrary constants and s is a positive constant.

Thus a similarity transformation between Eq. (1) and Eq. (3) can be obtained

$$u=\delta_0\alpha^{3/2}exp\left(\int\gamma\mathrm{d}z\right)\psi(\theta,\xi,\eta,\tau)exp(\mathrm{i}\phi), \tag{9}$$

where $\{\theta,\xi,\eta,\tau,\phi\}$ are determined by Eq. (6) with Eqs. (7)–(8).

Remark 1. To our knowledge, the similarity transformation (Eq. (9)) is a more general transformation. On the one hand, some results by many authors can be reproduced from it. We can cite two examples as follows. (i) If setting $M(z)=0$ in Eq. (7), we can derive a solution $\alpha=(1+h\int_0^z\rho\mathrm{d}z)^{-1}$. If further setting $b_j=c_j=0(j=1,2,3)$ and eliminating the constrained conditions (Eq. (8)), it is easy to verify that the similarity transformation obtained in Ref. [24] can be recovered by our similarity transformation (Eq. (9)). (ii) If setting $g_5(z)=\gamma(z)=k_5=0$, $k_1=\frac{1}{2},k_3=1$ and $e_1=e_2=e_3=0$, the transformation in Ref. [25] can be reproduced by Eq. (9). On the other hand, the Lie point symmetry of Eq. (9) under some special parameters can be derived by the transformation (Eq. (9)). For example, if setting $\rho=k_1,g_3(z)=k_3,g_5(z)=k_5,M(z)=0$ in Eq. (1), i.e., Eq. (1) is the same form as Eq. (3) besides using different variables, thus we can derive the finite symmetry group transformation from Eq. (9) and further obtain the corresponding Lie point symmetry.

3. Exact spatiotemporal self-similar solutions for (3+1)D CQNLSE

In this section, we first give some simple solutions of Eq. (3) in terms of Jacobi elliptic functions, which involves hyperbolic secant functions and hyperbolic tangent functions as special cases. Next we can easily write down the general solutions for Eq. (1) by the similarity transformation (Eq. (9)).

With the help of symbolic computation and some direct assumptions, one can obtain six types of traveling wave solutions for