Optics Communications 284 (2011) 4248–4253

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Effect of the surface integral in the torque equation of the electromagnetic angular momentum on the Faraday rotation

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ARTICLE INFO

Article history:
Received 3 January 2011
Received in revised form 29 April 2011
Accepted 13 May 2011
Available online 27 May 2011

Keywords:
Electromagnetic angular momentum
Torque equation
Faraday rotation
Surface integral
Volume integral

ABSTRACT

A circularly polarized plane wave of infinite transverse extent ($\delta=\infty$) has no spin angular momentum, while a realistic light does carry it. This paradox originates from the presence ($\delta=\infty$) and absence ($\delta\approx 0$) of the surface integral in the total angular momentum $\boldsymbol{J}$. The same holds for the torque equation of $d\boldsymbol{J}/dt$, so that δ is also connected with the relative Faraday rotation angle Θ_F/θ_F when a radius (a) of a cylindrical medium with optical activity is only a little larger than that (b) of light beam, where Θ_F is the Faraday rotation angle and θ_F is the intrinsic Faraday rotation angle of a medium. It is shown here that it is possible to estimate δ for a realistic light from the drastic variation in Θ_F/θ_F near $b/a=1$.

1. Introduction

The angular momentum (AM) carried by light can be characterized by the spin AM associated with circular polarization and the orbital AM associated with the spatial distribution of the wave. Both spin and orbital angular momenta of light beam have in fact been measured [1–4]. Theoretically, the spin and orbital angular momenta ($\boldsymbol{S}$ and $\boldsymbol{L}$) of the radiation fields have been defined explicitly for free space and an isotropic medium [5–8]. The spin AM arises even from the transverse plane electromagnetic waves, but the orbital AM never appears in the plane electromagnetic waves.

As is generally known, however, a circularly polarized plane wave of *infinite* transverse extent can have no spin AM [9]. However, only a quasiplane wave of *finite* transverse extent δ carries the spin AM whose direction is along the direction of propagation. As evident from the definition of the AM, the component of the AM in the direction of propagation must be zero, but it is non-zero actually. This paradox has been subject of discussion for a long time [9] and even recently [10–12]. Some ideas [8,10,13–15] were proposed to resolve this paradox, but it has not yet been settled. At present, one justifies these results by taking into account the fact that a detector placed in a plane wave causes gradients in this field [14]. The field can no longer be considered as a plane wave. In other words, any obstacle that absorbs the beam changes the electromagnetic field at the edges of the obstacle so that the field components in the direction of propagation are produced. Recently, another thought different from this was proposed by Stewart [15]. He took into account the effect of boundaries on the plane wave problem by decomposing the AM into three items of two volume integrals of a spin character and an orbital character and one surface integral and concluded that the contribution to the AM arises from the edges of the beam. This idea seems to be reasonable and consistent theoretically. In his paper [15], however, the contribution from the surface integral to the AM has not been expressed as a function of δ, where δ is the transverse extent of the plane wave. For this reason, it is shown here that the contribution from the surface integral to the AM of the plane wave can be expressed analytically as a function of δ, on the assumption that the intensity profile of the optical fields forms a flat plateau for $r\leq b$ and decreases according to exp $[-(r-b)/\delta]$ for $r>b$, where r is the radial distance from the center axis of beam and b is a radius of beam. The same is also applicable to the torque equation of AM. This torque equation links directly to the relative Faraday rotation angle (Θ_F/θ_F) expressed as a function of δ, where Θ_F is the Faraday rotation angle and θ_F is the intrinsic Faraday rotation angle of a medium. Of course, the transverse extent δ has not yet been measured for a realistic light, because there was no experimental procedure for measuring it. For this reason, we show here that it is possible theoretically to estimate δ from the measurement of the Faraday rotation angle when a is only a little larger than b, where a is a radius of an optically cylindrical medium. When this experiment was carried out successfully, the transverse extent of a realistic light is revealed so that the above paradox is resolved explicitly.

The purpose of this study is to clarify the relation between the surface integrals in the torque equation for AM and the transverse extent δ or the relative Faraday rotation angle Θ_F/θ_F, and to provide a new experimental procedure for measuring the transverse extent of a realistic light, resulting in the resolution of the traditional paradox.

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0030-4018/$ – see front matter
doi:10.1016/j.optcom.2011.05.031

2. Analysis

2.1. The definition of angular momentum for electromagnetic fields

The Maxwell equations for the macroscopic electromagnetic fields in a medium are given by [16]

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1}$$

$$\nabla\times\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \mathrm{I}, \tag{2}$$

$$\nabla\cdot\boldsymbol{D} = \rho, \tag{3}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{4}$$

It is assumed here that the medium is inhomogeneous and anisotropic but is free of dispersion. In these equations, the electric $\boldsymbol{D}$ and magnetic $\boldsymbol{B}$ induction fields are related to the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ as $D_{i=x,y,z}=\Sigma_{j=x,y,z}\varepsilon_{ij}E_j$ and $B_{i=x,y,z}=\Sigma_{j=x,y,z}\mu_{ij}H_j$ where ε_{ij} and μ_{ij} represent the permittivity and permeability tensors, $\boldsymbol{I}$ is the electric current density, and ρ is the charge density.

Let us consider an arbitrary volume τ inside the crystal filled by charges, described by the volume density ρ and current density $\boldsymbol{I}$. Because of the interaction with the electromagnetic field, the charges experience a total mechanical torque $\mathrm{d}\boldsymbol{L}_c/dt$ given by [17]

$$\frac{\mathrm{d}\boldsymbol{L}_c}{\mathrm{d}t} = \int_\tau \mathrm{d}v\,\boldsymbol{r}\times[\rho\boldsymbol{E} + (\boldsymbol{I}\times\boldsymbol{B})], \tag{5}$$

where $\boldsymbol{L}_c$ is the mechanical angular momentum of the (charged and neutral) particles. Substituting $\boldsymbol{I}$ and ρ from Eqs. (2) and (3) into Eq. (5), we get, after some manipulations,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}_c + \boldsymbol{J}) = \int_\tau \mathrm{d}v\{\boldsymbol{r}\times[\boldsymbol{E}(\nabla\cdot\boldsymbol{D}) - \boldsymbol{D}\times(\nabla\times\boldsymbol{E}) - \boldsymbol{B}\times(\nabla\times\boldsymbol{H})]\}, \tag{6}$$

where the angular momentum $\boldsymbol{J}$ of electromagnetic fields in a medium is defined as

$$\boldsymbol{J} = \int_\tau \mathrm{d}v[\boldsymbol{r}\times(\boldsymbol{D}\times\boldsymbol{B})] \tag{7}$$

in analogy to the interpretation of the linear momentum density $(\boldsymbol{D}\times\boldsymbol{B})$ in a medium [6,16]. This definition is valid at least when the medium is linear, but not necessarily isotropic, in its response [16]. However, we do not enter here into the well-known question of the correct definition of momentum in media.

2.2. The angular momenta of radiation fields and their torque equations

The electromagnetic fields can be separated into transverse and longitudinal fields, which have by definition a vanishing divergence and curl, respectively. The magnetic field is purely transverse, while the longitudinal electric induction field $\boldsymbol{D}_\parallel$ is given by the instantaneous Coulomb field arising from the charge density ρ, i.e., $\nabla\cdot\boldsymbol{D}_\parallel=\rho$. The transverse electric induction field $\boldsymbol{D}_\perp$ thus describes the radiation part, which contains in fact the only real dynamical degrees of freedom of the field. The symbols || and $\perp$ denote the components parallel and perpendicular to the optical axis, respectively. For $\boldsymbol{D}_\perp$, therefore, Eq. (3) may be rewritten as

$$\nabla\cdot\boldsymbol{D}_\perp = 0. \tag{8}$$

Generally, the radiation gage is defined as $\varphi=0$ and $\nabla\cdot\boldsymbol{A}=0$ in the gage transformation, where φ is the scalar potential and $\boldsymbol{A}$ is the vector potential[16]. This gage is applicable to the transverse part of the electromagnetic fields and is often used when the charge and current are absent [6,7]. In the radiation gage, therefore, the electric field $\boldsymbol{E}_\perp$ and magnetic induction field $\boldsymbol{B}$ are expressed by the vector potential $\boldsymbol{A}_\perp$ as [7]

$$\boldsymbol{E}_\perp = -\frac{\partial \boldsymbol{A}_\perp}{\partial t} \tag{9}$$

and

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}_\perp, \tag{10}$$

where $\boldsymbol{E}_\perp$ and $\boldsymbol{A}_\perp$ are the transverse components of $\boldsymbol{E}$ and $\boldsymbol{A}$. In other words, this is the same as the Coulomb gage for $\varphi=0$. The transverse part $\boldsymbol{A}_\perp$ of the vector potential is thus gage invariant. We will hereafter treat $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{A}$ as $\boldsymbol{D}_\perp$, $\boldsymbol{E}_\perp$ and $\boldsymbol{A}_\perp$, respectively. By substituting Eq. (10) into Eq. (7) and applying partial integration, the angular momentum $\boldsymbol{J}$ [5–7] defined previously for the transverse electromagnetic field is separated into three parts as

$$\boldsymbol{J} = \int_\tau(\boldsymbol{D}\times\boldsymbol{A})\mathrm{d}v + \int_\tau \Sigma_{i=x,y,z} D_i(\boldsymbol{r}\times\nabla)A_i\mathrm{d}v + \int_\Sigma(\boldsymbol{A}\times\boldsymbol{r})(\boldsymbol{D}\cdot\mathrm{d}\boldsymbol{s}) = \boldsymbol{S} + \boldsymbol{L} + S_\Sigma, \tag{11}$$

where the symbol ∇ is the gradient operator, and $\boldsymbol{S}$, $\boldsymbol{L}$ and $\boldsymbol{S}_\Sigma$ represent the volume integrals with spin and orbital characters and the surface integral, respectively, which are as follows,

$$\boldsymbol{S} = \int_\tau(\boldsymbol{D}\times\boldsymbol{A})\mathrm{d}v, \tag{12}$$

$$\boldsymbol{L} = \int_\tau \Sigma_{i=x,y,z} D_i(\mathbf{r}\times\nabla)A_i\mathrm{d}v \tag{13}$$

and

$$\boldsymbol{S}_\Sigma = \int_\Sigma(\boldsymbol{A}\times\boldsymbol{r})(\boldsymbol{D}\cdot\mathrm{d}\boldsymbol{s}). \tag{14}$$

When the radiation gage is employed, therefore, the angular momentum of $\boldsymbol{J}$ is gage independent. In this sense, the separation of $\boldsymbol{S}$ and $\boldsymbol{L}$ has a clear physical meaning [6,7], but we recognize that the identification of terms as spin and orbital momenta may not be unique in general. However, the orbital AM is ignored in this and subsequent subsections because we treat only the plane electromagnetic wave. When the electromagnetic field in an isotropic medium is composed of the plane wave of *infinite* extent, the surface integral of $\boldsymbol{S}_\Sigma$ can be transformed to the volume integral of $-\int_\tau(\boldsymbol{D}\times\boldsymbol{A})\mathrm{d}v$, so that $\boldsymbol{S}$ and $\boldsymbol{S}_\Sigma$ cancel out, resulting in $\boldsymbol{J}=0$ [9]. When the electromagnetic field has *infinitesimal* extent, however, the surface integral of $\boldsymbol{S}_\Sigma$ vanishes and only the volume integral of $\boldsymbol{S}$ survives, resulting in $\boldsymbol{J}=\boldsymbol{S}$. The surface integral thus varies significantly with changes in the magnitude of the transverse extent of the plane wave. Let us consider the plane waves of finite extent δ which form a light beam of radius b propagating along the z axis in a cylindrical medium of radius a, as shown in Fig. 1. Since it is difficult to calculate exactly the intensity profile of the optical fields, for simplicity, it is assumed here that the intensity profile of the optical fields forms a flat plateau for $r\le b$, while for $r\ge b$, it is expressed by a function of exp $[-(r-b)/\delta]$, as shown in Fig. 1(b), where r is the radial distance from the center axis of beam and $(r-b)$ is the distance from the edge of beam. It means that the intensity of the optical fields in the outside of beam core decreases exponentially and concentrically with an increase of $(r-b)$. When the intensity of the optical fields at the distance δ from the beam edge is assumed to lower down to $1/e$ of that at the beam edge, δ represents the average transverse extent of the optical fields protruded slightly from the core edge, as evident from the integral value of $\int_0^\infty e^{-r/\delta}\mathrm{d}r=\delta$. In practice, the intensity of the optical fields (including the vector potential $\boldsymbol{A}$) would probably decrease rapidly near the edge of beam in a complicated manner. However, even this simple intensity profile is