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Entanglement in Weisskopf–Wigner theory of atomic decay in free space

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ABSTRACT

In this paper, we use the Weisskopf–Wigner theory to study the entanglement in the state of the free-space radiation field produced from vacuum due to atomic decay. We show how bipartite entanglement is shared between different partitions of the radiation modes. We investigate the role played by the size of the partitions and their detuning with the decaying atom. The dynamics of the atom-field entanglement during the atomic decay is also briefly discussed. From this dynamics, we assert that such entanglement is the physical quantity that fix the statistical atomic decay time.

Entanglement plays a central role in quantum information science where it is seen as an important physical resource for information processing beyond the achievable with classical correlations [1–3]. Investigations in quantum phase transitions [4,5] and statistical mechanics have been realized from the point of view of entanglement [6]. In this approach, concepts like randomness, ensemble-averaging or time-averaging are not required. Instead, thermalization results from entanglement between system and environment. Connections between matter and quantum information theory have also been discussed [7,8]. Such studies have pointed out that entanglement seems to be important in other areas of physics besides pure quantum information. It is exactly in this scope that this work is introduced.

Considering entanglement as a legitimate physical quantity, this paper is intended to study a fundamental process namely the atomic spontaneous emission. It lies at the core of matter–radiation interaction. The successful description of atomic decay in free space is one of the remarkable achievements of the quantum theory of radiation [9–11]. Why does an excited atom decay? It is clear that an isolated atom would never decay from one excited state to another with lower energy because both are eigenstates of the system Hamiltonian. There must be some physical system to couple to the atom in order drive the electronic transition. This external agent is the free space electromagnetic field whose zero-point energy fluctuations are able to cause the atom to decay. In the language of quantum information theory, the entanglement between atom and field is then the responsible for the atomic decay. In what follows, entanglement is studied in the spontaneous emission phenomenon.

We are particularly interested in the entanglement properties of quantum fields. In this paper, the quantized field is the bosonic free-space continuous electromagnetic field. We will analyze the entanglement properties of this field after atomic decay. Entanglement in discretized bosonic fields have already been studied. In particular, the entanglement properties of the ground state and thermal states of this system is studied in detail in [12]. Discrete versions of real free Klein–Gordon fields have also been studied from the point of view of entanglement [13]. In this study, the relation between entanglement, entropy, and area for a specific harmonic lattice whose continuum limit lead to the Hamiltonian of the real Klein–Gordon field is analyzed in great detail. These papers triggered many others studies that also investigated entanglement in discretized quantum fields [14–17].

The starting point of the present work is the Weisskopf–Wigner theory of spontaneous emission which is now briefly presented [9–11]. In the rotating wave approximation, a two-level atom interacts with the free-space electromagnetic field according to the interaction picture Hamiltonian [10]

$$\widehat{H} = \hbar \sum_{\mathbf{k}} \left[g^*_{\mathbf{k}}(\mathbf{r}_0)\sigma_+ \hat{a}_{\mathbf{k}} e^{i(\omega-\nu_k)t} + \text{H.c.} \right], \tag{1}$$

where ω is the angular frequency of the atomic transition (excited state $|a\rangle$ and ground state $|b\rangle$), $\mathbf{r}_0$ is the position of the atom, $\hat{a}_{\mathbf{k}}$ is the

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annihilation operator for the field mode $\{\mathbf{k}\}$ (angular frequency v_k), and

$$g_{\mathbf{k}}(\mathbf{r}_0) = g_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}_0}, \tag{2}$$

where

$$g_{\mathbf{k}} = -\sqrt{\frac{\hbar v_k}{2\epsilon_0 V}} \frac{\wp_{ab}\cdot\hat{\epsilon}_{\mathbf{k}}}{\hbar}, \tag{3}$$

with V a quantization volume, ϵ_0 the electric permittivity of free space, $\wp_{ab}$ the dipole moment for the atomic transition, and $\hat{\epsilon}_{\mathbf{k}}$ the polarization vector of the mode $\{\mathbf{k}\}$. It will be assumed that initially the atom is in the excited state $|a\rangle$ and the field modes are in the vacuum $|0\rangle = |0,0,\ldots\rangle$. According to (1) the system evolved state will be

$$|\psi(t)\rangle = c_a(t)|a,0\rangle + \sum_{\mathbf{k}} c_{b,\mathbf{k}}(t)|b,1_{\mathbf{k}},\{0\}\rangle, \tag{4}$$

where $|1_{\mathbf{k}},\{0\}\rangle$ represents the field state with one photon in the mode $\{\mathbf{k}\}$ and the rest in the vacuum, and

$$c_a(t) = e^{-\Gamma t/2}, \tag{5}$$

$$c_{b,\mathbf{k}}(t) = g_{\mathbf{k}}(\mathbf{r}_0)\frac{1-e^{i(\omega-v_k)t-\Gamma t/2}}{(v_k-\omega)+i\Gamma/2}, \tag{6}$$

with Γ being the free-space atomic decay constant which is giving by

$$\Gamma = \frac{1}{4\pi\epsilon_0}\frac{4\omega^3\wp_{ab}^2}{3\hbar c^3}. \tag{7}$$

In order to obtain the above equations, it was considered that the intensity of the light associated with the emitted radiation is very centered about the atomic frequency ω. This is the essence of the Weisskopf–Wigner theory. In this theory, the free-space modes act as an immediate response reservoir, i.e., the atomic spontaneous emission is seen as a Markovian process.

Now, the entanglement content in the field state after spontaneous decay of the atom is studied in detail. This state is denoted $|\gamma_0\rangle$ and it is obtained from (4) by assuming $t \gg \Gamma^{-1}$

$$|\gamma_0\rangle = \sum_{\mathbf{k}} \frac{g_{\mathbf{k}}(\mathbf{r}_0)}{(v_k-\omega)+i\Gamma/2}|1_{\mathbf{k}},\{0\}\rangle. \tag{8}$$

It is worth noticing that the state $|\gamma_0\rangle$ is, from the point of view of quantum information science, a member of an important class of multipartite entangled states called generalized W states [18]. However, we must take care when using this state. Although state (8) is presented as a discrete summation over $\mathbf{k}$, any kind of calculation using it is to be done transforming it to an integral, i.e., an continuum of modes.

The state (8) represents all modes of the free-space radiation field, and it is a superposition of the different possibilities of distributing one photon (emitted by the atom) between the infinity of modes. Consequently, this is an entangled multipartite state whose bipartite entanglement between different partitions of radiation modes is now going to be investigated. There are many ways of partitioning the free-space modes in two partitions. We think it is physically appealing to choose one partition formed by a central mode with frequency $v_{\mathbf{q}}$ and modes distributed in the interval $(v_{\mathbf{q}}-\epsilon, v_{\mathbf{q}}+\epsilon)$ (let us call it partition A), and the other partition formed by the rest (partition B). This is an interesting physical choice since it allows us to check the effect of having $v_{\mathbf{q}}$ either near or far from resonance with the decaying two-level atom (frequency separation ω), and to check the importance of the size of the partitions via the parameter ϵ.

Since $|\gamma_0\rangle$ is a pure state, the appropriate entanglement measure between partitions A and B of the system is the entropy of entanglement $E = S(\rho_A)$, where $S(\rho_A) = -\mathrm{tr}[\rho_A \log_2(\rho_A)]$ is the von-Neumann entropy with the reduced state $\rho_A = \mathrm{tr}_B[\rho_{AB}]$. It must be emphasized that the entropy of entanglement is a entanglement monotone only if the global state is pure. Even though the pure field state $|\gamma_0\rangle$ is achieved only in the limit $t \gg \Gamma^{-1}$, we will see later on this paper that our results are still approximately valid for finite times. This broadens the applicability of our work. The reduced state for the partition A can be obtained from (8) by tracing out modes in partition B. One finds

$$\rho_A = \sum_{\mathbf{k}_j} |p_j|^2 |\{0\}_A\rangle\langle_A\{0\}| + \sum_{\mathbf{k}_m,\mathbf{k}_n} p_m p_n^* |1_{\mathbf{k}_m},\{0\}\rangle\langle 1_{\mathbf{k}_n},\{0\}|, \tag{9}$$

where $\mathbf{k}_j$ refers to a wave vector of some mode in the partition B, $\mathbf{k}_m$ ($\mathbf{k}_n$) to some mode in partition A, $|\{0\}_A\rangle$ to vacuum states of modes in partition A, $|1_{\mathbf{k}_{m(n)}},\{0\}\rangle$ means one photon in mode $\mathbf{k}_{m(n)}$ of partition A and vacuum for the rest of the modes in that partition, and

$$p_i = \frac{g_{\mathbf{k}_i}(\mathbf{r}_0)}{(v_{k_i}-\omega)+i\Gamma/2}. \tag{10}$$

The only non-zero eigenvalues of ρ_A are $\lambda_1 = \sum_{\mathbf{k}_j}|p_j|^2$ and $\lambda_2 = \sum_{\mathbf{k}_n}|p_n|^2$, where $\mathbf{k}_j$ refers to partition B and $\mathbf{k}_n$ to partition A. As mentioned before, the final results must be obtained by passing to the continuum. In spherical coordinates we have [10]

$$\sum_{\mathbf{k}} \to 2\frac{V}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2, \tag{11}$$

and then

$$\sum_{\mathbf{k}_j}|p_j|^2 \to \frac{\wp_{ab}}{6\pi^2\hbar\epsilon_0 c^3}\left[\int_0^{v_q-\epsilon}\frac{v^3 dv}{(v-\omega)^2+\Gamma^2/4} + \int_{v_q+\epsilon}^{\infty}\frac{v^3 dv}{(v-\omega)^2+\Gamma^2/4}\right]. \tag{12}$$

For consistency with the Weisskopf–Wigner used in the derivation of the state (8), we should again consider that v^3 varies little around $v_k = \omega$, what allows us now to replace v^3 by ω^3 in the above integrals as well as to extend the lower integration limit of the first integral to $-\infty$ [10]. Making this approximations one obtains

$$\sum_{\mathbf{k}_j}|p_j|^2 \to 1 - \frac{1}{\pi}\arctan\left[\frac{2}{\Gamma}(\epsilon + v_q - \omega)\right] - \frac{1}{\pi}\arctan\left[\frac{2}{\Gamma}(\epsilon - v_q + \omega)\right]. \tag{13}$$

Now, we sum the modes referring to partition A

$$\sum_{\mathbf{k}_n}|p_n|^2 \to \frac{\wp_{ab}}{6\pi^2\hbar\epsilon_0 c^3}\left[\int_{v_q-\epsilon}^{v_q+\epsilon}\frac{v^3 dv}{(v-\omega)^2+\Gamma^2/4}\right]. \tag{14}$$

Again, we replace v^3 by ω^3 (but leave the integration limits unaltered) to obtain

$$\sum_{\mathbf{k}_n}|p_n|^2 \to \frac{1}{\pi}\arctan\left[\frac{2}{\Gamma}(\epsilon + v_q - \omega)\right] + \frac{1}{\pi}\arctan\left[\frac{2}{\Gamma}(\epsilon - v_q + \omega)\right]. \tag{15}$$

It is important to look into normalization of (8) because we must end up with a physical state after performing the approximations. In fact, the normalization has been conserved since the sum of (13) with (15) is equal to one for any values of ϵ, ω and v_q. With (13) and (15), one can now easily obtain the entropy of entanglement $S = -\sum_{i=1}^{2}\lambda_i \log_2 \lambda_i$ and study the bipartite entanglement between partitions A and B. In general, two special features of the entanglement in the field modes should be highlighted, namely its dependence upon the size of the partitions and upon the detuning between the central frequency of partition A and the atom $\Delta = v_q - \omega$. From now on, we will use the dimensionless quantities $\tilde{\epsilon} \equiv \epsilon/\Gamma$ and $\widetilde{\Delta} \equiv \Delta/\Gamma$.