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# Master equations with memory for systems driven by classical noise

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A non-Markovian master equation is obtained for a two level atom driven by a phase noisy laser. The derivation is based on obtaining an equation for the density operator of the system averaged over the previous histories of the external noise. Averaging over the current value of the noise variable by means of the Zwanzig–Nakajima projection operator technique leads to a master equation with memory and a local-in-time master equation. The solutions to the resultant non-Markovian master equation, the structural properties of the equation, and the amenability of the equation to unravelling by the quantum trajectory method are all investigated.



## 1. Introduction

All physical systems, to a greater or lesser extent, are open, that is the system of interest interacts with its surrounding environment. This environment is often identified with the 'rest of the universe', in which case it is typically treated as a very large quantum system with many degrees of freedom in thermal equilibrium at some temperature – a thermal bath or reservoir. Less generally, the 'environment' can be a simple quantum system, e.g. a single qubit. In either case, the interaction between the system and its environment results in the system experiencing 'quantum noise' which shows up in the system exhibiting fluctuations, decoherence, and possibly irreversible dissipative dynamics. A further kind of open system is one in which the interaction with the external world can be modelled as deterministic or stochastic *classical* influences such as, for instance, a classical EM field interacting with a quantum mechanical atom. In either case, what is usually of interest is the state of the system alone. In the wholly quantum case, this state is given by a reduced density operator obtained by tracing over the environmental degrees of freedom. In the case of a classical *stochastic* external influence, a density operator description is obtained by taking an average over all the realizations of the associated classical stochastic process. The aim then is to derive the master equation for this system density operator $\hat{\rho}$, not only because the solution to this equation provides a full description of the system dynamics, but also because the structure of the master equation can be revealing of the character of this underlying dynamics, as well as providing physical interpretations of this dynamics, such as those implied by a quantum trajectory unravelling of the equation.

The master equation can be shown, for instance by the projection operator techniques of Zwanzig–Nakajima [12,13], to assume the form of an integro-differential equation:

$$\frac{d\hat{\rho}}{dt} = -i[\widehat{H}(t),\hat{\rho}] + \int_0^t \mathscr{K}(t,\tau)\hat{\rho}(\tau)\,d\tau, \qquad (1)$$

i.e. there appears a memory kernel $\mathscr{K}(t,\tau)$ which determines the extent to which the current state of the system depends on its past history.

An important distinguishing feature of the underlying dynamics is whether it is Markovian or not. Loosely speaking, the evolution is said to be Markovian if the future state of the system depends solely on its current state, and not on its earlier history. No physical system is truly Markovian, but provided the correlation between the system and environment becomes negligibly small over a time which is infinitesimal compared to all the other time scales of the system evolution, memory effects can be neglected: the Markov approximation. In effect the memory kernel $\mathscr{K}(t,\tau)$ is approximated by a delta function $\sim \delta(t-\tau)$, an approximation that is valid in very many cases of physical interest. The corresponding circumstance for a classically driven system is if the external influence is linearly coupled delta function correlated white noise. The result in either case is the much-studied and well-understood quantum Markovian master equation which assumes a particular form, the so-called Lindblad form given by [1,2]

$$\frac{d\hat{\rho}}{dt} = \mathscr{L}\hat{\rho} = -i[\widehat{H},\hat{\rho}] + \sum_m \gamma_m\left[\widehat{L}_m\hat{\rho}\widehat{L}_m^\dagger - \frac{1}{2}\left\{\widehat{L}_m^\dagger\widehat{L}_m,\hat{\rho}\right\}\right], \qquad (2)$$

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where the $\gamma_m$ are real and $\geqslant 0$. This equation preserves the trace and Hermiticity of the system density operator, and the solution defines a completely positive linear map $\mathscr{F}(t)\hat{\rho}(0) = \hat{\rho}(t)$. The Lindbald structure also implies that such a master equation can be unravelled as an ensemble of quantum trajectories, with the possibility of a concomitant measurement interpretation [3,4].

If the correlations between the system and environment persist sufficiently long for the Markov approximation not to be justified, as is the case in many physical situations of interest, e.g. in solid state systems, or in quantum Brownian motion [5], then the master equation would be expected to be non-Markovian, and assume the form of Eq. (1). Similarly, if the system is driven by external classical colored noise, the master equation for the averaged density operator should also be non-Markovian. Krzysztof Wódkiewicz (KW) contributed significantly to developing an understanding of non-Markovian systems in both senses, for instance in the study of quantum non-Markovian effects in resonance fluorescence [6], effects of classical phase noise in strong laser–atom interactions [7–9], or more recently in the study of memory effects in the non-Markovian versions of the Bloch equations [10,11]. In these latter works the structure and properties of the master equation itself and of the quantum evolution it describes, in particular the importance of completely positive evolution, was the focus of attention.

It is possible to push this result further using the time-convolutionless operator technique [14] which leads to a local-in-time first-order differential equation for $\hat{\rho}$ (see also [15]). The possible form of this master equation that preserves the Hermiticity and trace of $\hat{\rho}$, though not necessarily complete positivity is implicit in the work Gorini–Kossakowski–Sudarshan [2] and is given by

$$\frac{d\hat{\rho}}{dt} = \mathscr{L}(t)\hat{\rho} = -i[\widehat{H}(t), \hat{\rho}] + \sum_m \gamma_m(t)\left[\widehat{L}_m(t)\hat{\rho}\widehat{L}_m^\dagger(t) - \frac{1}{2}\left\{\widehat{L}_m^\dagger(t)\widehat{L}_m(t), \hat{\rho}\right\}\right], \tag{3}$$

i.e. the generator $\mathscr{L}(t)$ may depend on time, it may give rise to negative quantum jump probabilities when $\mathrm{Re}[\gamma_m(t)] < 0$ and hence does not yield an obvious quantum trajectory unravelling, it will not necessarily generate a completely positive map and finally, the equation will not necessarily be valid for all time due to a generic property of such equations: the generator $\mathscr{L}(t)$ may become singular at a finite time [16,17]. The overall structure of the master equation is very similar to that of the Markovian master equation, and has been referred to, and will be referred to here, as being of quasi-Lindblad form.

Even though non-Markov quantum systems play an increasingly important role in many areas of physics there is currently no fully developed theory of non-Markovian systems. Amongst other issues, one of the challenges in formulating a theory of non-Markovian open systems is the paucity of models for which it is even possible to derive an exact master equation [18], though in recent times more powerful techniques have been developed which show promise of changing this state of affairs [19,20]. But it is systems driven by classical stochastic influences that have been a fertile source of non-Markovian master equations, as can be seen in the recent works of KW cited above, and, in the work of others, including, e.g. [21–23].

In this paper, we present in Section 2 an approach to deriving the master equation for a class of systems driven in a possibly non-linear fashion by classical noise. The method used is closely related to the stochastic Liouville master equation approach developed by Kubo [24,25]. It is shown that the master equation for the system obtained by averaging over this noise will, in general, be non-Markovian. The master equation for an example of interest for many years in quantum optics is then derived in Section 3, this being the master equation for a two level atom driven by a phase noisy laser. In Section 3.2 the master equation of the form of Eq. (1) is derived, and the local-in-time master equation derived and discussed in Section 3.3 where it will be shown to exhibit many of the characteristics mentioned above of non-Markovian master equations: quasi-Lindblad structure, negative damping, and singular behavior in time. However, it will be shown that the master equation generates a completely positive map. A brief discussion of possible quantum trajectory unravellings of this equation are addressed in Section 3.4, followed by a conclusion.

## 2. Derivation of general equation

The aim here is to derive a general way of constructing the master equation for a system driven by external classical stochastic process $\phi(t)$, a simple one-dimensional random walk or Wiener process. Before constructing this master equation, it is useful to consider how the probability distribution $P(\phi, t)$ of $\phi$ can be constructed. If we assume that in a small time interval $\Delta t$, there is equal probability $R\Delta t$ of $\phi$ increasing or decreasing by an amount $\Delta\phi$ then we have

$$P(\phi, t + \Delta t) = P(\phi, t)(1 - R\Delta t) + \frac{1}{2}RP(\phi - \Delta\phi, t) + \frac{1}{2}RP(\phi + \Delta\phi, t). \tag{4}$$

In the limit of $\Delta t \to 0$, and $\Delta\phi \to 0$ and $R \to \infty$ such that

$$D = \frac{1}{2}R(\Delta\phi)^2 \tag{5}$$

is the constant diffusion rate, this leads to the well-known diffusion equation

$$\frac{\partial P(\phi, t)}{\partial t} = D\frac{\partial^2 P(\phi, t)}{\partial \phi^2}. \tag{6}$$

For the quantum system under consideration, stochasticity enters by virtue of the system Hamiltonian $\widehat{H}(\phi(t))$ being dependent on the stochastic variable $\phi$. Let $\hat{\rho}(\phi, t)$ be the density operator for the system at time $t$ for the current value of the random variable $\phi(t)$. It will be normalized to unity, i.e.

$$\mathrm{Tr}[\hat{\rho}(\phi, t)] = 1. \tag{7}$$

The density operator $\hat{\rho}(\phi, t)$ will be the density operator obtained by averaging over all the previous history of evolution that lead to the random variable $\phi$ having the current value $\phi$. For a small time interval $\Delta t$, the density operator $\rho(\phi, t + \Delta t)$ will be made up from three contributions depending on whether the $\hat{\rho}(\phi, t)$ evolves freely with no change in $\phi$, or else there have occurred 'jumps' $\hat{\rho}(\phi \pm \Delta\phi, t) \to \hat{\rho}(\phi, t + \Delta t)$. The probability of these alternatives are $(1 - R\Delta t)$ and $\frac{1}{2}R\Delta t$ respectively. The states $\hat{\rho}(\phi, t)$ and $\hat{\rho}(\phi \pm \Delta\phi, t)$ have to be weighted by the probabilities $P(\phi, t)$ and $P(\phi \pm \Delta\phi, t)$ of the system being in the states $\hat{\rho}(\phi, t)$ and $\hat{\rho}(\phi \pm \Delta\phi, t)$ at time $t$. Putting this all together, and normalizing the state yields

$$\begin{aligned}\hat{\rho}(\phi, t + \Delta t) = &\bigg\{(1 + \mathscr{L}(\phi)\Delta t)(1 - R\Delta t)P(\phi, t)\hat{\rho}(\phi, t) \\ &+ \frac{1}{2}R\Delta t P(\phi - \Delta\phi, t)\hat{\rho}(\phi - \Delta\phi, t) \\ &+ \frac{1}{2}R\Delta t P(\phi + \Delta\phi, t)\hat{\rho}(\phi + \Delta\phi, t)\bigg\} \\ &\div \left\{(1 - R\Delta t)P(\phi, t) + \frac{1}{2}R\Delta t[P(\phi - \Delta\phi, t) + P(\phi + \Delta\phi, t)]\right\},\end{aligned} \tag{8}$$