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Beams of electromagnetic radiation carrying angular momentum: The Riemann–Silberstein vector and the classical–quantum correspondence

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Abstract

All beams of electromagnetic radiation are made of photons. Therefore, it is important to find a precise relationship between the classical properties of the beam and the quantum characteristics of the photons that make a particular beam. It is shown that this relationship is best expressed in terms of the Riemann–Silberstein vector – a complex combination of the electric and magnetic field vectors – that plays the role of the photon wave function. The Whittaker representation of this vector in terms of a single complex function satisfying the wave equation greatly simplifies the analysis. Bessel beams, exact Laguerre–Gauss beams, and other related beams of electromagnetic radiation can be described in a unified fashion. The appropriate photon quantum numbers for these beams are identified. Special emphasis is put on the angular momentum of a single photon and its connection with the angular momentum of the beam.

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1. Introduction

"*The notion of photon as a quantum of electromagnetic energy, like the notion of atom as an elementary unit of matter, permits a number of interpretations.*" With these words Bruce Shore begins the section entitled "What is a photon" in his comprehensive monograph [1] and then he goes on to present a few possible answers to this question. We hope that our contribution to this special issue adds a little to the neverending discussion on the nature of photons.

Electromagnetic radiation, especially in the optical range, is produced and used most often in the form of beams. There is a variety of mathematical models to describe such beams: the Bessel beams, the Hermite–Gauss beams, the Laguerre–Gauss beams, and also the focus wave modes. The mathematical representations of these beams are the exact, or approximate, solutions of the classical Maxwell equations. In recent years, there were many experiments that exhibited the influence of the orbital momentum on the beam properties (phase dislocations, optical vortices) and on the interaction of beams with matter (particle trapping, optical tweezers and spanners). As a rule, orbital angular momentum leads to vortices and electromagnetic beams with vortices, as we shown recently [2,3], are interesting because they can guide charged particles. The fundamental theoretical and experimental papers on the optical angular momentum were recently reprinted in a collection [4]. In all these papers, the theoretical description of beams carrying angular momentum is given

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in terms of the classical solutions of the Maxwell equations and sometimes separately in terms of photons. However, no unifying principle is given that would connect these two points of view in a precise manner. The purpose of this work is to fill this gap. We shall show that the notion of the photon wave function appears in both of these descriptions and that it provides a very convenient concept to unify the two points of view. A very useful mathematical tool in this analysis is the Riemann–Silberstein vector [5–11]. Applications of the RS vector to many physical problems were recently reviewed in a very thorough paper by Keller [12]. The Whittaker representation [13] of this vector greatly simplifies the calculations since then the vector field is described by a single function. We shall not consider here the beams described by *approximate* solutions of Maxwell equations obtained in the paraxial approximation [14]. The analysis of such solutions in terms of photons would be rather awkward – the notion of an approximate photon does not make sense.

There is some overlap between our final conclusions and those obtained recently by Járegui and Hacyan [15] even though our methods are completely different. They rely on the standard description of the quantized electromagnetic field based on the vector potential. We follow the methods developed in our earlier works on quantum electrodynamics [9] and photon wave functions [11] where the Riemann–Silberstein vector plays the central role.

2. Succinct description of the electromagnetic field

A natural tool in the analysis of the solutions of the Maxwell equations, in both classical and quantum theories, is the Riemann–Silberstein (RS) vector $\boldsymbol{F}$

$$\boldsymbol{F} = \sqrt{\frac{\epsilon_0}{2}}(\boldsymbol{E} + \mathrm{i}c\boldsymbol{B}). \tag{1}$$

The physical significance of the RS vector has been recognized by Silberstein [6] who observed that important characteristics of the electromagnetic field (energy density, Poynting vector, Maxwell stress tensor) are bilinear products of the components of this vector. The total energy

$$E = \int \mathrm{d}^3r\boldsymbol{F}^* \cdot \boldsymbol{F}, \tag{2}$$

the total momentum (i.e. the integral of the Poynting vector divided by c)

$$\boldsymbol{P} = \frac{-\mathrm{i}}{c}\int \mathrm{d}^3r\boldsymbol{F}^* \times \boldsymbol{F}, \tag{3}$$

and the total angular momentum

$$\boldsymbol{M} = \frac{-\mathrm{i}}{c}\int \mathrm{d}^3r(\boldsymbol{r} \times (\boldsymbol{F}^* \times \boldsymbol{F})) \tag{4}$$

look very much like the quantum-mechanical expectation values and we shall show later that this fact has a deeper meaning.

The convenience of using the RS vector has been recognized by Bateman [7], who was the first to analyze with its help various solutions of the Maxwell equations. Kramers [16] used the RS vector to formulate the canonical theory of the electromagnetic field and Power [17] stressed the usefulness of this vector in the description of circularly polarized waves. The complex RS vector carries exactly the same information as two real field vectors but its use significantly simplifies the mathematical analysis. In particular, the two pairs of real Maxwell equations written in terms of $\boldsymbol{F}$ reduce to just one pair of complex Maxwell equations

$$\partial_t\boldsymbol{F}(\boldsymbol{r},t) = -\mathrm{i}c\nabla \times \boldsymbol{F}(\boldsymbol{r},t), \quad \nabla \cdot \boldsymbol{F}(\boldsymbol{r},t) = 0. \tag{5}$$

The RS vector can be expressed in the following form [18]

$$\boldsymbol{F}(\boldsymbol{r},t) = \nabla \times \left(\frac{\mathrm{i}}{c}\partial_t\boldsymbol{Z}(\boldsymbol{r},t) + \nabla \times \boldsymbol{Z}(\boldsymbol{r},t)\right), \tag{6}$$

where $\boldsymbol{Z}(\boldsymbol{r},t)$ is a complex vector field (a unified form of the two Hertz vector potentials) satisfying the d'Alembert equation

$$\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\boldsymbol{Z}(\boldsymbol{r},t) = 0. \tag{7}$$

In the description of beams, it is convenient to choose the vector $\boldsymbol{Z}$ in the direction of propagation $\boldsymbol{Z}(\boldsymbol{r},t) = (0,0,1)$ $\chi(\boldsymbol{r},t)$. In this way, we obtain the following representation of the RS vector in terms of one complex function $\chi(\boldsymbol{r},t)$

$$F_x(\boldsymbol{r},t) = \left(\partial_x\partial_z + \frac{\mathrm{i}}{c}\partial_y\partial_t\right)\chi(\boldsymbol{r},t), \tag{8a}$$

$$F_y(\boldsymbol{r},t) = \left(\partial_y\partial_z - \frac{\mathrm{i}}{c}\partial_x\partial_t\right)\chi(\boldsymbol{r},t), \tag{8b}$$

$$F_z(\boldsymbol{r},t) = -(\partial_x^2 + \partial_y^2)\chi(\boldsymbol{r},t). \tag{8c}$$

We shall refer to these formulas as the Whittaker representation. A century ago Whittaker discovered [13] that an electromagnetic field obeying the Maxwell equation can be described by two real functions. By separating Eqs. (8) into the real and imaginary part, we recover the original Whittaker's formulas. The Whittaker representation (8) of the RS vector is the simplest but it is not unique. For example, by choosing the complex Hertz vector $\boldsymbol{Z}$ in the form $\boldsymbol{Z}(\boldsymbol{r},t) = (1,\mathrm{i},0)\chi(\boldsymbol{r},t)$ we obtain a different expression for $\boldsymbol{F}$ that will be used in Section 9.

The representation of the solutions of the Maxwell equations in terms of a single complex function (or two real functions for that matter) satisfying the d'Alembert equation greatly simplifies the analysis. This has been noticed already by Stratton [8] in his derivation of the formulas for the Bessel beams. Since there are no auxiliary conditions (like the vanishing of the divergence) imposed on χ, this single function describes true degrees of freedom of the electromagnetic field. Sometimes, following Whittaker, we shall call χ a "scalar" solution of the d'Alembert equation. However, strictly speaking this terminology is not justified because χ has fairly complicated transformation properties, quite different from those of a scalar field.